$$\begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix} = \begin{pmatrix} 0 & s_{12} & 0 \\ s_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & s_{13} \\ 0 & 0 & 0 \\ s_{31} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s_{23} \\ 0 & s_{32} & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} s_{11} & 0 & 0 \\ 0 & -s_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -s_{33} & 0 \\ 0 & 0 & s_{33} \end{pmatrix}$$

where the last two tensors are seen to be equivalent to simple shear states by comparison of cases (*a*) and (*b*) in Problem 2.26. Also note that since $s_{ii} = 0$, $-s_{11} - s_{33} = s_{22}$.

2.29. Determine the principal deviator stress values for the stress tensor

$$\sigma_{ij} = \begin{pmatrix} 10 & -6 & 0 \\ -6 & 10 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The deviator of σ_{ij} is $s_{ij} = \begin{pmatrix} 3 & -6 & 0 \\ -6 & 3 & 0 \\ 0 & 0 & -6 \end{pmatrix}$ and its principal values may be determined from the determinant

$$\begin{vmatrix} 3-s & -6 & 0 \\ -6 & 3-s & 0 \\ 0 & 0 & -6-s \end{vmatrix} = (-6-s)(s+3)(s-9) = 0$$

Thus $s_{\text{I}} = 9$, $s_{\text{II}} = -3$, $s_{\text{III}} = -6$. The same result is obtained by first calculating the principal stress values of σ_{ij} and then using *(2.71)*. For σ_{ij}, as the reader should show, $\sigma_{\text{I}} = 16$, $\sigma_{\text{II}} = 4$, $\sigma_{\text{III}} = 1$ and hence $s_{\text{I}} = 16 - 7 = 9$, $s_{\text{II}} = 4 - 7 = -3$, $s_{\text{III}} = 1 - 7 = -6$.

2.30. Show that the second invariant of the stress deviator is given in terms of its principal stress values by $\text{II}_{\Sigma_D} = (s_{\text{I}}s_{\text{II}} + s_{\text{II}}s_{\text{III}} + s_{\text{III}}s_{\text{I}})$, or by the alternative form $\text{II}_{\Sigma_D} = -\frac{1}{2}(s_{\text{I}}^2 + s_{\text{II}}^2 + s_{\text{III}}^2)$.

In terms of the principal deviator stresses the characteristic equation of the deviator stress tensor is given by the determinant

$$\begin{vmatrix} s_{\text{I}}-s & 0 & 0 \\ 0 & s_{\text{II}}-s & 0 \\ 0 & 0 & s_{\text{III}}-s \end{vmatrix} = (s_{\text{I}}-s)(s_{\text{II}}-s)(s_{\text{III}}-s) = 0$$

$$= s^3 + (s_{\text{I}}s_{\text{II}} + s_{\text{II}}s_{\text{III}} + s_{\text{III}}s_{\text{I}})s - s_{\text{I}}s_{\text{II}}s_{\text{III}}$$

Hence from (2.72), $\text{II}_{\Sigma_D} = (s_{\text{I}}s_{\text{II}} + s_{\text{II}}s_{\text{III}} + s_{\text{III}}s_{\text{I}})$. Since $s_{\text{I}} + s_{\text{II}} + s_{\text{III}} = 0$,

$$\text{II}_{\Sigma_D} = \tfrac{1}{2}(2s_{\text{I}}s_{\text{II}} + 2s_{\text{II}}s_{\text{III}} + 2s_{\text{III}}s_{\text{I}} - (s_{\text{I}} + s_{\text{II}} + s_{\text{III}})^2) = -\tfrac{1}{2}(s_{\text{I}}^2 + s_{\text{II}}^2 + s_{\text{III}}^2)$$

MISCELLANEOUS PROBLEMS

2.31. Prove that for any symmetric tensor such as the stress tensor σ_{ij}, the transformed tensor σ'_{ij} in any other coordinate system is also symmetric.

From *(2.27)*, $\sigma'_{ij} = a_{ip}a_{jq}\sigma_{pq} = a_{jq}a_{ip}\sigma_{qp} = \sigma'_{ji}$.

2.32. At the point P the principal stresses are such that $2\sigma_{II} = \sigma_I + \sigma_{III}$. Determine the unit normal n_i for the plane upon which $\sigma_N = \sigma_{II}$ and $\sigma_S = (\sigma_I - \sigma_{III})/4$.

From (*2.33*), $\sigma_N = n_1^2\sigma_I + n_2^2(\sigma_I + \sigma_{III})/2 + n_3^2\sigma_{III} = (\sigma_I + \sigma_{III})/2$; and since $n_1^2 + n_2^2 + n_3^2 = 1$, these equations may be combined to yield $n_1 = n_3$. Next from (*2.47*),

$$\sigma_S^2 = n_1^2\sigma_I^2 + n_2^2(\sigma_I + \sigma_{III})^2/4 + n_3^2\sigma_{III}^2 - (\sigma_I + \sigma_{III})^2/4 = (\sigma_I - \sigma_{III})^2/16$$

Substituting $n_1 = n_3$ and $n_2^2 - 1 = -n_1^2 - n_3^2 = -2n_1^2$ into this equation and solving for n_1, the direction cosines are found to be $n_1 = 1/2\sqrt{2}$, $n_2 = \sqrt{3}/2$, $n_3 = 1/2\sqrt{2}$. The reader should apply these results to the stress tensor $\sigma_{ij} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}$.

2.33. Show that the stress tensor σ_{ij} may be decomposed into a spherical and deviatoric part in one and only one way.

Assume two decompositions, $\sigma_{ij} = \delta_{ij}\lambda + s_{ij} = \delta_{ij}\lambda^* + s_{ij}^*$ with $s_{ii} = 0$ and $s_{ii}^* = 0$. Then $\sigma_{ii} = 3\lambda = 3\lambda^*$, so $\lambda = \lambda^*$; and from $\lambda\delta_{ij} + s_{ij} = \lambda\delta_{ij} + s_{ij}^*$ it follows that $s_{ij} = s_{ij}^*$.

2.34. Prove that the principal stress values are real if $\mathbf{\Sigma}$ is real and symmetric.

For real values of the stress components the stress invariants are real and hence the coefficients in (*2.38*) are all real. Thus by the theory of equations one root (principal stress) is real. Let $\sigma_{(3)}$ be this root and consider a set of primed axes x_i' of which x_3' is in the direction of $\sigma_{(3)}$. With respect to such axes the characteristic equation is given by the determinant

$$\begin{vmatrix} \sigma_{11}' - \sigma & \sigma_{12}' & 0 \\ \sigma_{21}' & \sigma_{22}' - \sigma & 0 \\ 0 & 0 & \sigma_{(3)} - \sigma \end{vmatrix} = 0$$

or $(\sigma_{(3)} - \sigma)[(\sigma_{11}' - \sigma)(\sigma_{22}' - \sigma) - \sigma_{12}'^2] = 0$. Since the discriminant of the quadratic in square brackets $D = (\sigma_{11}' + \sigma_{22}')^2 - 4[\sigma_{11}'\sigma_{22}' - (\sigma_{12}')^2] = (\sigma_{11}' - \sigma_{22}')^2 + 4\sigma_{12}'^2 > 0$, the remaining roots must be real.

2.35. Use the method of Lagrangian multipliers to show that the extremal values (maximum and minimum) of the normal stress σ_N correspond to principal values.

From (*2.33*), $\sigma_N = \sigma_{ij}n_in_j$ with $n_in_i = 1$. Thus in analogy with (*2.51*) construct the function $H = \sigma_N - \lambda n_in_i$ for which $\partial H/\partial n_i = 0$. Then

$$\begin{aligned} \frac{\partial H}{\partial n_p} &= \sigma_{ij}n_{i,p}n_j + \sigma_{ij}n_in_{j,p} - 2\lambda n_{i,p}n_i \\ &= \sigma_{ij}\delta_{ip}n_j + \sigma_{ij}n_i\delta_{jp} - 2\lambda\delta_{ip}n_i \\ &= \sigma_{pj}n_j + \sigma_{ip}n_i - 2\lambda\delta_{ip}n_i = 2(\sigma_{pi} - \lambda\delta_{ip})n_i = 0 \end{aligned}$$

which is equivalent to equation (*2.36*) defining principal stress directions.

2.36. Assume that the stress components σ_{ij} are derivable from a symmetric tensor field ϕ_{ij} by the relationship $\sigma_{ij} = \epsilon_{ipq}\epsilon_{jmn}\phi_{qn,pm}$. Show that in the absence of body forces the equilibrium equations (*2.23*) are satisfied.

Using the results of Problem 1.58, the stress components are given by

$$\sigma_{ij} = \delta_{ij}(\phi_{qq,pp} - \phi_{qp,qp}) + \phi_{pi,pj} + \phi_{jp,pi} - \phi_{pp,ji} - \phi_{ji,pp}$$

or explicitly

$$\begin{aligned} \sigma_{11} &= \phi_{33,22} + \phi_{22,33} & \sigma_{12} &= \sigma_{21} = -\phi_{33,21} \\ \sigma_{22} &= \phi_{11,33} + \phi_{33,11} & \sigma_{23} &= \sigma_{32} = -\phi_{11,23} \\ \sigma_{33} &= \phi_{22,11} + \phi_{11,22} & \sigma_{31} &= \sigma_{13} = -\phi_{22,13} \end{aligned}$$

Substituting these values into $\sigma_{ij,j} = 0$,

$$\sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} = \phi_{33,221} + \phi_{22,331} - \phi_{33,212} - \phi_{22,133} = 0$$
$$\sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} = -\phi_{33,211} + \phi_{11,332} + \phi_{33,112} - \phi_{11,233} = 0$$
$$\sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} = -\phi_{22,131} - \phi_{11,232} + \phi_{22,113} + \phi_{11,223} = 0$$

2.37. Show that, as is asserted in Section 2.9, the normal to the Cauchy stress quadric at the point whose position vector is **r** is parallel to the stress vector $t_i^{(\hat{n})}$.

Let the quadric surface be given in the form $\phi = \sigma_{ij}\zeta_i\zeta_j \pm k^2 = 0$. The normal at any point is then $\nabla\phi$ or $\partial\phi/\partial\zeta_i = \phi_{,i}$. Hence $\phi_{,p} = \sigma_{ij}\delta_{ip}\zeta_j + \sigma_{ij}\zeta_i\delta_{jp} = 2\sigma_{pi}\zeta_i$. Now since $\zeta_i = rn_i$, this becomes $2\sigma_{pi}rn_i$ or $2r(\sigma_{pi}n_i) = 2t_p^{(\hat{n})}$.

2.38. At a point P the stress tensor referred to axes $Ox_1x_2x_3$ is given by

$$\sigma_{ij} = \begin{pmatrix} 15 & -10 & 0 \\ -10 & 5 & 0 \\ 0 & 0 & 20 \end{pmatrix}$$

If new axes $Ox_1'x_2'x_3'$ are chosen by a rotation about the origin for which the transformation matrix is $[a_{ij}] = \begin{bmatrix} 3/5 & 0 & -4/5 \\ 0 & 1 & 0 \\ 4/5 & 0 & 3/5 \end{bmatrix}$, determine the traction vectors on each of the primed coordinate planes by projecting the traction vectors of the original axes onto the primed directions. In this way determine σ'_{ij}. Check the result by the transformation formula (*2.27*).

From (*2.6*) and the identity $t_j^{(\hat{e}_i)} = \sigma_{ij}$ (*2.7*), the traction vectors on the unprimed coordinate axes are $\mathbf{t}^{(\hat{e}_1)} = 15\hat{\mathbf{e}}_1 - 10\hat{\mathbf{e}}_2$, $\mathbf{t}^{(\hat{e}_2)} = -10\hat{\mathbf{e}}_1 + 5\hat{\mathbf{e}}_2$, $\mathbf{t}^{(\hat{e}_3)} = 20\hat{\mathbf{e}}_3$ which correspond to the rows of the stress tensor. Projecting these vectors onto the primed axes by the vector form of (*2.12*), $\mathbf{t}^{(\hat{n})} = n_1\mathbf{t}^{(\hat{e}_1)} + n_2\mathbf{t}^{(\hat{e}_2)} + n_3\mathbf{t}^{(\hat{e}_3)}$, gives

$$\mathbf{t}^{(\hat{e}_1')} = \tfrac{3}{5}(15\hat{\mathbf{e}}_1 - 10\hat{\mathbf{e}}_2) - \tfrac{4}{5}(20\hat{\mathbf{e}}_3) = 9\hat{\mathbf{e}}_1 - 6\hat{\mathbf{e}}_2 - 16\hat{\mathbf{e}}_3$$

which by the transformation of the unit vectors becomes

$$\mathbf{t}^{(\hat{e}_1')} = 9(\tfrac{3}{5}\hat{\mathbf{e}}_1' + \tfrac{4}{5}\hat{\mathbf{e}}_3') - 6\hat{\mathbf{e}}_2' - 16(-\tfrac{4}{5}\hat{\mathbf{e}}_1' + \tfrac{3}{5}\hat{\mathbf{e}}_3') = 91\hat{\mathbf{e}}_1'/5 - 6\hat{\mathbf{e}}_2' - 12\hat{\mathbf{e}}_3'/5$$

Likewise
$$\mathbf{t}^{(\hat{e}_2')} = -6\hat{\mathbf{e}}_1' + 5\hat{\mathbf{e}}_2' - 8\hat{\mathbf{e}}_3'$$

and
$$\mathbf{t}^{(\hat{e}_3')} = -12\hat{\mathbf{e}}_1'/5 - 8\hat{\mathbf{e}}_2' + 84\hat{\mathbf{e}}_3'/5$$

so that
$$\sigma'_{ij} = \begin{pmatrix} 91/5 & -6 & -12/5 \\ -6 & 5 & -8 \\ -12/5 & -8 & 84/5 \end{pmatrix}$$

By (*2.27*),
$$\sigma'_{ij} = \begin{bmatrix} 3/5 & 0 & -4/5 \\ 0 & 1 & 0 \\ 4/5 & 0 & 3/5 \end{bmatrix}\begin{bmatrix} 15 & -10 & 0 \\ -10 & 5 & 0 \\ 0 & 0 & 20 \end{bmatrix}\begin{bmatrix} 3/5 & 0 & 4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \end{bmatrix}$$
$$= \begin{bmatrix} 9 & -6 & -16 \\ -10 & 5 & 0 \\ 12 & -8 & 12 \end{bmatrix}\begin{bmatrix} 3/5 & 0 & 4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \end{bmatrix} = \begin{bmatrix} 91/5 & -6 & -12/5 \\ -6 & 5 & -8 \\ -12/5 & -8 & 84/5 \end{bmatrix}$$

2.39. Show that the second invariant of the deviator stress tensor II_{Σ_D} is related to the octahedral shear stress by the equation $\sigma_{OCT} = \sqrt{-\frac{2}{3}II_{\Sigma_D}}$.

From Problem 2.22, $\sigma_{OCT} = \frac{1}{3}\sqrt{(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2}$, and because $\sigma_I = \sigma_M + s_I$, $\sigma_{II} = \sigma_M + s_{II}$, etc.,

$$\sigma_{OCT} = \frac{1}{3}\sqrt{(s_I - s_{II})^2 + (s_{II} - s_{III})^2 + (s_{III} - s_I)^2}$$
$$= \frac{1}{3}\sqrt{2(s_I^2 + s_{II}^2 + s_{III}^2) - 2(s_I s_{II} + s_{II}s_{III} + s_{III}s_I)}$$

Also $s_I + s_{II} + s_{III} = 0$ and so $(s_I + s_{II} + s_{III})^2 = 0$ or

$$s_I^2 + s_{II}^2 + s_{III}^2 = -2(s_I s_{II} + s_{II}s_{III} + s_{III}s_I)$$

Hence $$\sigma_{OCT} = \frac{1}{3}\sqrt{-6(s_I s_{II} + s_{II}s_{III} + s_{III}s_I)} = \sqrt{-\frac{2}{3}II_{\Sigma_D}}$$

2.40. The state of stress throughout a body is given by the stress tensor

$$\sigma_{ij} = \begin{pmatrix} 0 & Cx_3 & 0 \\ Cx_3 & 0 & -Cx_1 \\ 0 & -Cx_1 & 0 \end{pmatrix}$$

where C is an arbitrary constant. (*a*) Show that the equilibrium equations are satisfied if body forces are zero. (*b*) At the point $P(4,-4,7)$ calculate the stress vector on the plane $2x_1 + 2x_2 - x_3 = -7$, and on the sphere $(x_1)^2 + (x_2)^2 + (x_3)^2 = (9)^2$. (*c*) Determine the principal stresses, maximum shear stresses and principal deviator stresses at P. (*d*) Sketch the Mohr's circles for the state of stress at P.

(*a*) Substituting directly into (*2.24*) from σ_{ij}, the equilibrium equations are satisfied identically.

(*b*) From Problem 1.2, the unit normal to the plane $2x_1 + 2x_2 - x_3 = -7$ is $\hat{n} = \frac{2}{3}\hat{e}_1 + \frac{2}{3}\hat{e}_2 - \frac{1}{3}\hat{e}_3$. Thus from (*2.12*) the stress vector on the plane at P is

$$t^{(\hat{n})} = (\tfrac{2}{3}\hat{e}_1 + \tfrac{2}{3}\hat{e}_2 - \tfrac{1}{3}\hat{e}_3)\cdot(7C\hat{e}_1\hat{e}_2 + 7C\hat{e}_2\hat{e}_1 - 4C\hat{e}_2\hat{e}_3 - 4C\hat{e}_3\hat{e}_2)$$
$$= C(\tfrac{14}{3}\hat{e}_2 + \tfrac{14}{3}\hat{e}_1 - \tfrac{8}{3}\hat{e}_3 + \tfrac{4}{3}\hat{e}_2) = \tfrac{1}{3}C(14\hat{e}_1 + 18\hat{e}_2 - 8\hat{e}_3)$$

The normal to the sphere $x_ix_i = (9)^2$ at P is $n_i = \phi_{,i}$ with $\phi = x_ix_i - 81$, or $\hat{n} = \frac{4}{9}\hat{e}_1 - \frac{4}{9}\hat{e}_2 + \frac{7}{9}\hat{e}_3$. In the matrix form of (*2.14*) the stress vector at P is

$$[4/9, -4/9, 7/9]\begin{bmatrix} 0 & 7C & 0 \\ 7C & 0 & -4C \\ 0 & -4C & 0 \end{bmatrix} = [-28C/9, 0, 16C/9]$$

(*c*) From (*2.37*), for principal stresses σ,

$$\begin{vmatrix} -\sigma & 7 & 0 \\ 7 & -\sigma & -4 \\ 0 & -4 & -\sigma \end{vmatrix} = \sigma(\sigma^2 - 65) = 0; \text{ hence}$$

$\sigma_I = \sqrt{65}$, $\sigma_{II} = 0$, $\sigma_{III} = -\sqrt{65}$. The maximum shear stress value is given by (*2.54b*) as $\sigma_S = (\sigma_{III} - \sigma_I)/2 = \pm\sqrt{65}$. Since the mean normal stress at P is $\sigma_M = (\sigma_I + \sigma_{II} + \sigma_{III})/3 = 0$, the principal deviator stresses are the same as the principal stresses.

(*d*) The Mohr's circles are shown in Fig. 2-33.

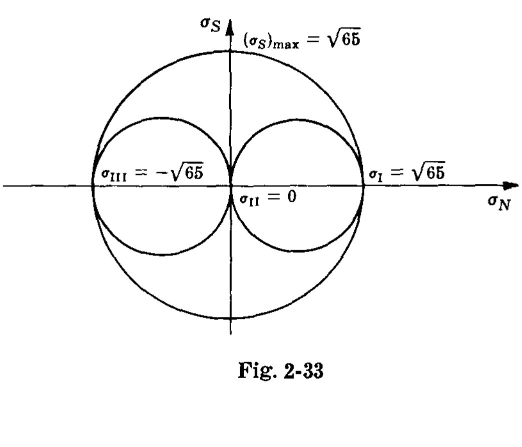


Fig. 2-33

Supplementary Problems

2.41. At point P the stress tensor is $\sigma_{ij} = \begin{pmatrix} 14 & 7 & -7 \\ 7 & 21 & 0 \\ -7 & 0 & 35 \end{pmatrix}$. Determine the stress vector on the plane at P parallel to plane (a) BGE, (b) $BGFC$ of the small parallelepiped shown in Fig. 2-34.

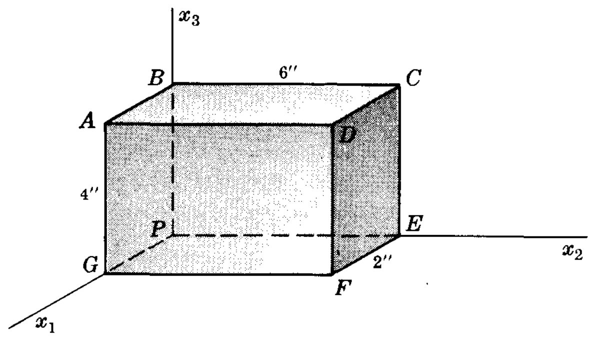


Fig. 2-34

Ans. (a) $\mathbf{t}^{(\hat{\mathbf{n}})} = 11\hat{\mathbf{e}}_1 + 12\hat{\mathbf{e}}_2 + 9\hat{\mathbf{e}}_3$, (b) $\mathbf{t}^{(\hat{\mathbf{n}})} = (21\hat{\mathbf{e}}_1 + 14\hat{\mathbf{e}}_2 + 21\hat{\mathbf{e}}_3)/\sqrt{5}$

2.42. Determine the normal and shear stress components on the plane $BGFC$ of Problem 2.41.
Ans. $\sigma_N = 63/5$, $\sigma_S = 37.7/5$

2.43. The principal stresses at point P are $\sigma_I = 12$, $\sigma_{II} = 3$, $\sigma_{III} = -6$. Determine the stress vector and its normal component on the octahedral plane at P.
Ans. $\mathbf{t}^{(\hat{\mathbf{n}})} = (12\hat{\mathbf{e}}_1 + 3\hat{\mathbf{e}}_2 - 6\hat{\mathbf{e}}_3)/\sqrt{3}$, $\sigma_N = 3$

2.44. Determine the principal stress values for

$$(a)\quad \sigma_{ij} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad (b)\quad \sigma_{ij} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

and show that both have the same principal directions.
Ans. (a) $\sigma_I = 2$, $\sigma_{II} = \sigma_{III} = -1$, (b) $\sigma_I = 4$, $\sigma_{II} = \sigma_{III} = 1$

2.45. Decompose the stress tensor $\sigma_{ij} = \begin{pmatrix} 3 & -10 & 0 \\ -10 & 0 & 30 \\ 0 & 30 & -27 \end{pmatrix}$ into its spherical and deviator parts and determine the principal deviator stresses. *Ans.* $s_I = 31$, $s_{II} = 8$, $s_{III} = -39$

2.46. Show that the normal component of the stress vector on the octahedral plane is equal to one third the first invariant of the stress tensor.

2.47. The stress tensor at a point is given as $\sigma_{ij} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & \sigma_{22} & 1 \\ 2 & 1 & 0 \end{pmatrix}$ with σ_{22} unspecified. Determine σ_{22} so that the stress vector on some plane at the point will be zero. Give the unit normal for this traction-free plane.
Ans. $\sigma_{22} = 1$, $\hat{\mathbf{n}} = (\hat{\mathbf{e}}_1 - 2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3)/\sqrt{6}$

2.48. Sketch the Mohr's circles and determine the maximum shear stress for each of the following stress states:

$$(a)\quad \sigma_{ij} = \begin{pmatrix} \tau & \tau & 0 \\ \tau & \tau & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (b)\quad \sigma_{ij} = \begin{pmatrix} \tau & 0 & 0 \\ 0 & -\tau & 0 \\ 0 & 0 & -2\tau \end{pmatrix}$$

Ans. (*a*) $\sigma_S = \tau$, (*b*) $\sigma_S = 3\tau/2$

2.49. Use the result given in Problem 1.58, page 39, together with the stress transformation law (*2.27*), page 50, to show that $\epsilon_{ijk}\epsilon_{pqm}\sigma_{ip}\sigma_{jq}\sigma_{km}$ is an invariant.

2.50. In a continuum, the stress field is given by the tensor

$$\sigma_{ij} = \begin{pmatrix} x_1^2 x_2 & (1-x_2^2)x_1 & 0 \\ (1-x_2^2)x_1 & (x_2^3 - 3x_2)/3 & 0 \\ 0 & 0 & 2x_3^2 \end{pmatrix}$$

Determine (*a*) the body force distribution if the equilibrium equations are to be satisfied throughout the field, (*b*) the principal stress values at the point $P(a, 0, 2\sqrt{a})$, (*c*) the maximum shear stress at P, (*d*) the principal deviator stresses at P.

Ans. (*a*) $b_3 = -4x_3$, (*b*) $a, -a, 8a$, (*c*) $\pm 4.5a$, (*d*) $-11a/3, -5a/3, 16a/3$

Chapter 3

Deformation and Strain

3.1 PARTICLES AND POINTS

In the kinematics of continua, the meaning of the word "point" must be clearly understood since it may be construed to refer either to a "point" in space, or to a "point" of a continuum. To avoid misunderstanding, the term "point" will be used exclusively to designate a location in fixed space. The word "particle" will denote a small volumetric element, or "material point", of a continuum. In brief, a *point* is a place in space, a *particle* is a small part of a material continuum.

3.2 CONTINUUM CONFIGURATION. DEFORMATION AND FLOW CONCEPTS

At any instant of time t, a continuum having a volume V and bounding surface S will occupy a certain region R of physical space. The identification of the particles of the continuum with the points of the space it occupies at time t by reference to a suitable set of coordinate axes is said to specify the *configuration* of the continuum at that instant.

The term *deformation* refers to a change in the shape of the continuum between some initial (undeformed) configuration and a subsequent (deformed) configuration. The emphasis in deformation studies is on the initial and final configurations. No attention is given to intermediate configurations or to the particular sequence of configurations by which the deformation occurs. By contrast, the word *flow* is used to designate the continuing state of motion of a continuum. Indeed, a configuration history is inherent in flow investigations for which the specification of a time-dependent velocity field is given.

3.3 POSITION VECTOR. DISPLACEMENT VECTOR

In Fig. 3-1 the undeformed configuration of a material continuum at time $t = 0$ is shown together with the deformed configuration of the same continuum at a later time $t = t$. For the present development it is useful to refer the initial and final configurations to separate coordinate axes as in the figure.

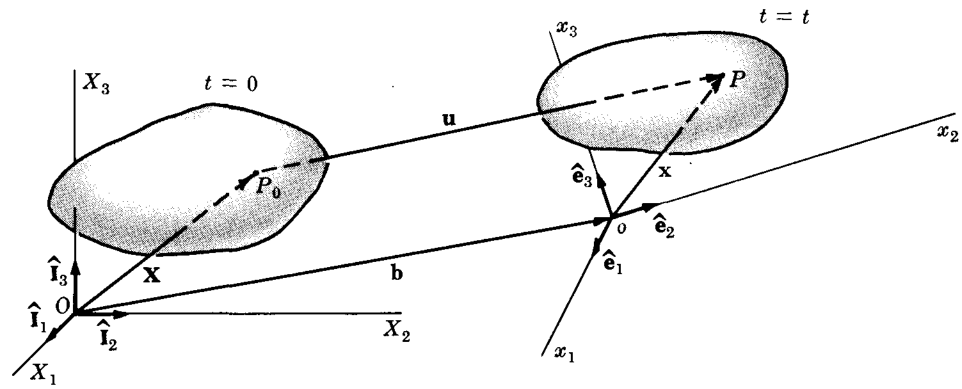


Fig. 3-1

Accordingly, in the initial configuration a representative particle of the continuum occupies a point P_0 in space and has the *position vector*

$$\mathbf{X} = X_1\hat{\mathbf{I}}_1 + X_2\hat{\mathbf{I}}_2 + X_3\hat{\mathbf{I}}_3 = X_K\hat{\mathbf{I}}_K \tag{3.1}$$

with respect to the rectangular Cartesian axes $OX_1X_2X_3$. Upper-case letters are used as indices in (*3.1*) and will appear as such in several equations that follow, but their use as summation indices is restricted to this section. In the remainder of the book upper-case subscripts or superscripts serve as labels only. Their use here is to emphasize the connection of certain expressions with the coordinates $(X_1X_2X_3)$, which are called the *material coordinates*. In the deformed configuration the particle originally at P_0 is located at the point P and has the position vector

$$\mathbf{x} = x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3\hat{\mathbf{e}}_3 = x_i\hat{\mathbf{e}}_i \tag{3.2}$$

when referred to the rectangular Cartesian axes $ox_1x_2x_3$. Here lower-case letters are used as subscripts to identify with the coordinates $(x_1x_2x_3)$ which give the current position of the particle and are frequently called the *spatial coordinates*.

The relative orientation of the material axes $OX_1X_2X_3$ and the spatial axes $ox_1x_2x_3$ is specified through direction cosines α_{kK} and α_{Kk}, which are defined by the dot products of unit vectors as

$$\hat{\mathbf{e}}_k \cdot \hat{\mathbf{I}}_K = \hat{\mathbf{I}}_K \cdot \hat{\mathbf{e}}_k = \alpha_{kK} = \alpha_{Kk} \tag{3.3}$$

No summation is implied by the indices in these expressions since k and K are distinct indices. Inasmuch as Kronecker deltas are designated by the equations $\hat{\mathbf{I}}_K \cdot \hat{\mathbf{I}}_P = \delta_{KP}$ and $\hat{\mathbf{e}}_k \cdot \hat{\mathbf{e}}_p = \delta_{kp}$, the *orthogonality conditions* between spatial and material axes take the form

$$\alpha_{Kk}\alpha_{Kp} = \alpha_{kK}\alpha_{pK} = \delta_{kp}; \quad \alpha_{Kp}\alpha_{Mp} = \alpha_{pK}\alpha_{pM} = \delta_{KM} \tag{3.4}$$

In Fig. 3-1 the vector $\mathbf{u}$ joining the points P_0 and P (the initial and final positions, respectively, of the particle), is known as the *displacement vector*. This vector may be expressed as

$$\mathbf{u} = u_k\hat{\mathbf{e}}_k \tag{3.5}$$

or alternatively as

$$\mathbf{U} = U_K\hat{\mathbf{I}}_K \tag{3.6}$$

in which the components U_K and u_k are interrelated through the direction cosines α_{kK}. From (*1.89*) the unit vector $\hat{\mathbf{e}}_k$ is expressed in terms of the material base vectors $\hat{\mathbf{I}}_K$ as

$$\hat{\mathbf{e}}_k = \alpha_{kK}\hat{\mathbf{I}}_K \tag{3.7}$$

Therefore substituting (*3.7*) into (*3.5*),

$$\mathbf{u} = u_k(\alpha_{kK}\hat{\mathbf{I}}_K) = U_K\hat{\mathbf{I}}_K = \mathbf{U} \tag{3.8}$$

from which

$$U_K = \alpha_{kK}u_k \tag{3.9}$$

Since the direction cosines α_{kK} are constants, the components of the displacement vector are observed from (*3.9*) to obey the law of transformation of first-order Cartesian tensors, as they should.

The vector $\mathbf{b}$ in Fig. 3-1 serves to locate the origin o with respect to O. From the geometry of the figure,

$$\mathbf{u} = \mathbf{b} + \mathbf{x} - \mathbf{X} \tag{3.10}$$

Very often in continuum mechanics it is possible to consider the coordinate systems $OX_1X_2X_3$ and $ox_1x_2x_3$ superimposed, with $\mathbf{b} \equiv \mathbf{0}$, so that (3.10) becomes

$$\mathbf{u} = \mathbf{x} - \mathbf{X} \tag{3.11}$$

In Cartesian component form this equation is given by the general expression

$$u_k = x_k - \alpha_{kK} X_K \tag{3.12}$$

However, for superimposed axes the unit triads of base vectors for the two systems are identical, which results in the direction cosine symbols α_{kK} becoming Kronecker deltas. Accordingly, (3.12) reduces to

$$u_k = x_k - X_k \tag{3.13}$$

in which only lower-case subscripts appear. In the remainder of this book, unless specifically stated otherwise, the material and spatial axes are assumed *superimposed* and hence only lower-case indices will be used.

3.4 LAGRANGIAN AND EULERIAN DESCRIPTIONS

When a continuum undergoes deformation (or flow), the particles of the continuum move along various paths in space. This motion may be expressed by equations of the form

$$x_i = x_i(X_1, X_2, X_3, t) = x_i(\mathbf{X}, t) \quad \text{or} \quad \mathbf{x} = \mathbf{x}(\mathbf{X}, t) \tag{3.14}$$

which give the present location x_i of the particle that occupied the point $(X_1X_2X_3)$ at time $t = 0$. Also, (3.14) may be interpreted as a mapping of the initial configuration into the current configuration. It is assumed that such a mapping is one-to-one and continuous, with continuous partial derivatives to whatever order is required. The description of motion or deformation expressed by (3.14) is known as the *Lagrangian* formulation.

If, on the other hand, the motion or deformation is given through equations of the form

$$X_i = X_i(x_1, x_2, x_3, t) = X_i(\mathbf{x}, t) \quad \text{or} \quad \mathbf{X} = \mathbf{X}(\mathbf{x}, t) \tag{3.15}$$

in which the independent variables are the coordinates x_i and t, the description is known as the *Eulerian* formulation. This description may be viewed as one which provides a tracing to its original position of the particle that now occupies the location (x_1, x_2, x_3). If (3.15) is a continuous one-to-one mapping with continuous partial derivatives, as was also assumed for (3.14), the two mappings are the unique inverses of one another. A necessary and sufficient condition for the inverse functions to exist is that the Jacobian

$$J = \left| \frac{\partial x_i}{\partial X_j} \right| \tag{3.16}$$

should not vanish.

As a simple example, the Lagrangian description given by the equations

$$\begin{aligned} x_1 &= X_1 + X_2(e^t - 1) \\ x_2 &= X_1(e^{-t} - 1) + X_2 \\ x_3 &= X_3 \end{aligned} \tag{3.17}$$

has the inverse Eulerian formulation,

$$\begin{aligned} X_1 &= \frac{-x_1 + x_2(e^t - 1)}{1 - e^t - e^{-t}} \\ X_2 &= \frac{x_1(e^{-t} - 1) - x_2}{1 - e^t - e^{-t}} \\ X_3 &= x_3 \end{aligned} \tag{3.18}$$

3.5 DEFORMATION GRADIENTS. DISPLACEMENT GRADIENTS

Partial differentiation of (*3.14*) with respect to X_j produces the tensor $\partial x_i/\partial X_j$ which is called the *material deformation gradient.* In symbolic notation, $\partial x_i/\partial X_j$ is represented by the dyadic

$$\mathbf{F} = \mathbf{x}\nabla_{\mathbf{X}} \equiv \frac{\partial \mathbf{x}}{\partial X_1}\hat{\mathbf{e}}_1 + \frac{\partial \mathbf{x}}{\partial X_2}\hat{\mathbf{e}}_2 + \frac{\partial \mathbf{x}}{\partial X_3}\hat{\mathbf{e}}_3 \tag{3.19}$$

in which the differential operator $\nabla_{\mathbf{X}} = \frac{\partial}{\partial X_i}\hat{\mathbf{e}}_i$ is applied from the right (as shown explicitly in the equation). The matrix form of $\mathbf{F}$ serves to further clarify this property of the operator $\nabla_{\mathbf{X}}$ when it appears as the consequent of a dyad. Thus

$$\mathcal{F} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\left[\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \frac{\partial}{\partial X_3}\right] = \begin{bmatrix} \partial x_1/\partial X_1 & \partial x_1/\partial X_2 & \partial x_1/\partial X_3 \\ \partial x_2/\partial X_1 & \partial x_2/\partial X_2 & \partial x_2/\partial X_3 \\ \partial x_3/\partial X_1 & \partial x_3/\partial X_2 & \partial x_3/\partial X_3 \end{bmatrix} = [\partial x_i/\partial X_j] \tag{3.20}$$

Partial differentiation of (*3.15*) with respect to x_j produces the tensor $\partial X_i/\partial x_j$ which is called the *spatial deformation gradient.* This tensor is represented by the dyadic

$$\mathbf{H} = \mathbf{X}\nabla_{\mathbf{x}} \equiv \frac{\partial \mathbf{X}}{\partial x_1}\hat{\mathbf{e}}_1 + \frac{\partial \mathbf{X}}{\partial x_2}\hat{\mathbf{e}}_2 + \frac{\partial \mathbf{X}}{\partial x_3}\hat{\mathbf{e}}_3 \tag{3.21}$$

having a matrix form

$$\mathcal{H} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right] = \begin{bmatrix} \partial X_1/\partial x_1 & \partial X_1/\partial x_2 & \partial X_1/\partial x_3 \\ \partial X_2/\partial x_1 & \partial X_2/\partial x_2 & \partial X_2/\partial x_3 \\ \partial X_3/\partial x_1 & \partial X_3/\partial x_2 & \partial X_3/\partial x_3 \end{bmatrix} = [\partial X_i/\partial x_j] \tag{3.22}$$

The material and spatial deformation tensors are interrelated through the well-known chain rule for partial differentiation,

$$\frac{\partial x_i}{\partial X_j}\frac{\partial X_j}{\partial x_k} = \frac{\partial X_i}{\partial x_j}\frac{\partial x_j}{\partial X_k} = \delta_{ik} \tag{3.23}$$

Partial differentiation of the displacement vector u_i with respect to the coordinates produces either the *material displacement gradient* $\partial u_i/\partial X_j$, or the *spatial displacement gradient* $\partial u_i/\partial x_j$. From (*3.13*), which expresses u_i as a difference of coordinates, these tensors are given in terms of the deformation gradients as the material gradient

$$\frac{\partial u_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j} - \delta_{ij} \quad \text{or} \quad \mathbf{J} \equiv \mathbf{u}\nabla_{\mathbf{X}} = \mathbf{F} - \mathbf{I} \tag{3.24}$$

and the spatial gradient

$$\frac{\partial u_i}{\partial x_j} = \delta_{ij} - \frac{\partial X_i}{\partial x_j} \quad \text{or} \quad \mathbf{K} \equiv \mathbf{u}\nabla_{\mathbf{x}} = \mathbf{I} - \mathbf{H} \tag{3.25}$$

In the usual manner, the matrix forms of $\mathbf{J}$ and $\mathbf{K}$ are respectively

$$\mathcal{J} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\left[\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \frac{\partial}{\partial X_3}\right] = \begin{bmatrix} \partial u_1/\partial X_1 & \partial u_1/\partial X_2 & \partial u_1/\partial X_3 \\ \partial u_2/\partial X_1 & \partial u_2/\partial X_2 & \partial u_2/\partial X_3 \\ \partial u_3/\partial X_1 & \partial u_3/\partial X_2 & \partial u_3/\partial X_3 \end{bmatrix} = [\partial u_i/\partial X_j] \tag{3.26}$$

and

$$\mathcal{K} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right] = \begin{bmatrix} \partial u_1/\partial x_1 & \partial u_1/\partial x_2 & \partial u_1/\partial x_3 \\ \partial u_2/\partial x_1 & \partial u_2/\partial x_2 & \partial u_2/\partial x_3 \\ \partial u_3/\partial x_1 & \partial u_3/\partial x_2 & \partial u_3/\partial x_3 \end{bmatrix} = [\partial u_i/\partial x_j] \tag{3.27}$$

3.6 DEFORMATION TENSORS. FINITE STRAIN TENSORS

In Fig. 3-2 the initial (undeformed) and final (deformed) configurations of a continuum are referred to the superposed rectangular Cartesian coordinate axes $OX_1X_2X_3$ and $ox_1x_2x_3$. The neighboring particles which occupy points P_0 and Q_0 before deformation, move to points P and Q respectively in the deformed configuration.

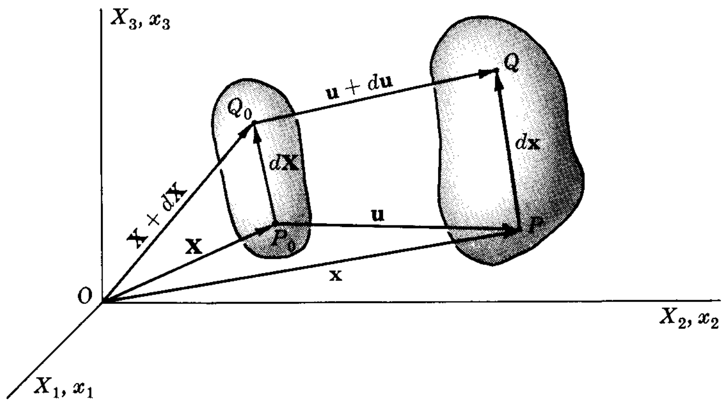


Fig. 3-2

The square of the differential element of length between P_0 and Q_0 is

$$(dX)^2 = d\mathbf{X}\cdot d\mathbf{X} = dX_i\,dX_i = \delta_{ij}\,dX_i\,dX_j \tag{3.28}$$

From (3.15), the distance differential dX_i is seen to be

$$dX_i = \frac{\partial X_i}{\partial x_j}dx_j \qquad \text{or} \qquad d\mathbf{X} = \mathbf{H}\cdot d\mathbf{x} \tag{3.29}$$

so that the squared length $(dX)^2$ in (3.28) may be written

$$(dX)^2 = \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}dx_i\,dx_j = C_{ij}\,dx_i\,dx_j \qquad \text{or} \qquad (dX)^2 = d\mathbf{x}\cdot\mathbf{C}\cdot d\mathbf{x} \tag{3.30}$$

in which the second-order tensor

$$C_{ij} = \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j} \qquad \text{or} \qquad \mathbf{C} = \mathbf{H}_c\cdot\mathbf{H} \tag{3.31}$$

is known as *Cauchy's deformation tensor.*

In the deformed configuration, the square of the differential element of length between P and Q is

$$(dx)^2 = d\mathbf{x}\cdot d\mathbf{x} = dx_i\,dx_i = \delta_{ij}\,dx_i\,dx_j \tag{3.32}$$

From (3.14) the distance differential here is

$$dx_i = \frac{\partial x_i}{\partial X_j}dX_j \qquad \text{or} \qquad d\mathbf{x} = \mathbf{F}\cdot d\mathbf{X} \tag{3.33}$$

so that the squared length $(dx)^2$ in (3.32) may be written

$$(dx)^2 = \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j}dX_i\,dX_j = G_{ij}\,dX_i\,dX_j \qquad \text{or} \qquad (dx)^2 = d\mathbf{X}\cdot\mathbf{G}\cdot d\mathbf{X} \tag{3.34}$$

in which the second-order tensor

$$G_{ij} = \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} \qquad \text{or} \qquad \mathbf{G} = \mathbf{F}_c\cdot\mathbf{F} \tag{3.35}$$

is known as *Green's deformation tensor.*

The difference $(dx)^2 - (dX)^2$ for two neighboring particles of a continuum is used as the *measure of deformation* that occurs in the neighborhood of the particles between the initial and final configurations. If this difference is identically zero for all neighboring particles of a continuum, a *rigid displacement* is said to occur. Using (*3.34*) and (*3.28*), this difference may be expressed in the form

$$(dx)^2 - (dX)^2 = \left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right) dX_i\, dX_j = 2L_{ij}\, dX_i\, dX_j$$

or

$$(dx)^2 - (dX)^2 = d\mathbf{X}\cdot(\mathbf{F}_c\cdot\mathbf{F} - \mathbf{I})\cdot d\mathbf{X} = d\mathbf{X}\cdot 2\mathbf{L}_G\cdot d\mathbf{X} \qquad (3.36)$$

in which the second-order tensor

$$L_{ij} = \frac{1}{2}\left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right) \quad \text{or} \quad \mathbf{L}_G = \tfrac{1}{2}(\mathbf{F}_c\cdot\mathbf{F} - \mathbf{I}) \qquad (3.37)$$

is called the *Lagrangian* (or Green's) *finite strain tensor.*

Using (*3.32*) and (*3.30*), the same difference may be expressed in the form

$$(dx)^2 - (dX)^2 = \left(\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right) dx_i\, dx_j = 2E_{ij}\, dx_i\, dx_j$$

or

$$(dx)^2 - (dX)^2 = d\mathbf{x}\cdot(\mathbf{I} - \mathbf{H}_c\cdot\mathbf{H})\cdot d\mathbf{x} = d\mathbf{x}\cdot 2\mathbf{E}_A\cdot d\mathbf{x} \qquad (3.38)$$

in which the second-order tensor

$$E_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right) \quad \text{or} \quad \mathbf{E}_A = \tfrac{1}{2}(\mathbf{I} - \mathbf{H}_c\cdot\mathbf{H}) \qquad (3.39)$$

is called the *Eulerian* (or Almansi's) *finite strain tensor.*

An especially useful form of the Lagrangian and Eulerian finite strain tensors is that in which these tensors appear as functions of the displacement gradients. Thus if $\partial x_i/\partial X_j$ from (*3.24*) is substituted into (*3.37*), the result after some simple algebraic manipulations is the Lagrangian finite strain tensor in the form

$$L_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right) \quad \text{or} \quad \mathbf{L}_G = \tfrac{1}{2}(\mathbf{J} + \mathbf{J}_c + \mathbf{J}_c\cdot\mathbf{J}) \qquad (3.40)$$

In the same manner, if $\partial X_i/\partial x_j$ from (*3.25*) is substituted into (*3.39*), the result is the Eulerian finite strain tensor in the form

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) \quad \text{or} \quad \mathbf{E}_A = \tfrac{1}{2}(\mathbf{K} + \mathbf{K}_c - \mathbf{K}_c\cdot\mathbf{K}) \qquad (3.41)$$

The matrix representations of (*3.40*) and (*3.41*) may be written directly from (*3.26*) and (*3.27*) respectively.

3.7 SMALL DEFORMATION THEORY. INFINITESIMAL STRAIN TENSORS

The so-called *small deformation theory* of continuum mechanics has as its basic condition the requirement that the displacement gradients be small compared to unity. The fundamental measure of deformation is the difference $(dx)^2 - (dX)^2$, which may be expressed in terms of the displacement gradients by inserting (*3.40*) and (*3.41*) into (*3.36*) and (*3.38*) respectively. If the displacement gradients are small, the finite strain tensors in (*3.36*) and (*3.38*) reduce to infinitesimal strain tensors, and the resulting equations represent small deformations.

In (3.40), if the displacement gradient components $\partial u_i/\partial X_j$ are each small compared to unity, the product terms are negligible and may be dropped. The resulting tensor is the *Lagrangian infinitesimal strain tensor,* which is denoted by

$$l_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \quad \text{or} \quad \mathbf{L} = \tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{X}} + \nabla_{\mathbf{X}}\mathbf{u}) = \tfrac{1}{2}(\mathbf{J}+\mathbf{J}_c) \qquad (3.42)$$

Likewise for $\partial u_i/\partial x_j \ll 1$ in (3.41), the product terms may be dropped to yield the *Eulerian infinitesimal strain tensor,* which is denoted by

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{or} \quad \mathbf{E} = \tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{x}} + \nabla_{\mathbf{x}}\mathbf{u}) = \tfrac{1}{2}(\mathbf{K}+\mathbf{K}_c) \qquad (3.43)$$

If both the displacement gradients and the displacements themselves are small, there is very little difference in the material and spatial coordinates of a continuum particle. Accordingly the material gradient components $\partial u_i/\partial X_j$ and spatial gradient components $\partial u_i/\partial x_j$ are very nearly equal, so that the Eulerian and Lagrangian infinitesimal strain tensors may be taken as equal. Thus

$$l_{ij} = \epsilon_{ij} \quad \text{or} \quad \mathbf{L} = \mathbf{E} \qquad (3.44)$$

if both the displacements and displacement gradients are sufficiently small.

3.8 RELATIVE DISPLACEMENTS. LINEAR ROTATION TENSOR. ROTATION VECTOR

In Fig. 3-3 the displacements of two neighboring particles are represented by the vectors $u_i^{(P_0)}$ and $u_i^{(Q_0)}$ (see also Fig. 3-2). The vector

$$du_i = u_i^{(Q_0)} - u_i^{(P_0)} \quad \text{or} \quad d\mathbf{u} = \mathbf{u}^{(Q_0)} - \mathbf{u}^{(P_0)} \qquad (3.45)$$

is called the *relative displacement vector* of the particle originally at Q_0 with respect to the particle originally at P_0. Assuming suitable continuity conditions on the displacement field, a Taylor series expansion for $u_i^{(P_0)}$ may be developed in the neighborhood of P_0. Neglecting higher-order terms in this expansion, the relative displacement vector can be written as

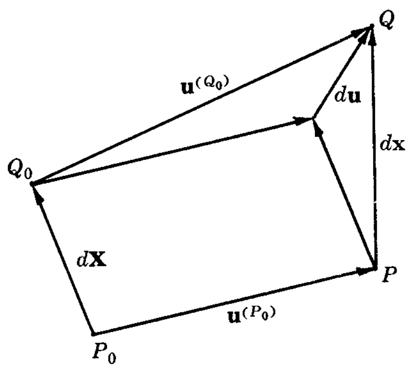


Fig. 3-3

$$du_i = (\partial u_i/\partial X_j)_{P_0}\,dX_j \quad \text{or} \quad d\mathbf{u} = (\mathbf{u}\nabla_{\mathbf{X}})_{P_0}\cdot d\mathbf{X} \qquad (3.46)$$

Here the parentheses on the partial derivatives are to emphasize the requirement that the derivatives are to be evaluated at point P_0. These derivatives are actually the components of the material displacement gradient. Equation (3.46) is the Lagrangian form of the relative displacement vector.

It is also useful to define the *unit relative displacement vector* du_i/dX in which dX is the magnitude of the differential distance vector dX_i. Accordingly if ν_i is a unit vector in the direction of dX_i so that $dX_i = \nu_i\,dX$, then

$$\frac{du_i}{dX} = \frac{\partial u_i}{\partial X_j}\frac{dX_j}{dX} = \frac{\partial u_i}{\partial X_j}\nu_j \quad \text{or} \quad \frac{d\mathbf{u}}{dX} = \mathbf{u}\nabla_{\mathbf{X}}\cdot\hat{\boldsymbol{\nu}} = \mathbf{J}\cdot\hat{\boldsymbol{\nu}} \qquad (3.47)$$

Since the material displacement gradient $\partial u_i/\partial X_j$ may be decomposed uniquely into a symmetric and an antisymmetric part, the relative displacement vector du_i may be written as

$$du_i = \left[\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right)\right] dX_j$$

or

$$d\mathbf{u} = [\tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{X}} + \nabla_{\mathbf{X}}\mathbf{u}) + \tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{X}} - \nabla_{\mathbf{X}}\mathbf{u})] \cdot d\mathbf{X} \tag{3.48}$$

The first term in the square brackets in (*3.48*) is recognized as the linear Lagrangian strain tensor l_{ij}. The second term is known as the *linear Lagrangian rotation tensor* and is denoted by

$$W_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right) \quad \text{or} \quad \mathbf{W} = \tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{X}} - \nabla_{\mathbf{X}}\mathbf{u}) \tag{3.49}$$

In a displacement for which the strain tensor l_{ij} is identically zero in the vicinity of point P_0, the relative displacement at that point will be an infinitesimal rigid body rotation. This infinitesimal rotation may be represented by the *rotation vector*

$$w_i = \tfrac{1}{2}\epsilon_{ijk}W_{kj} \quad \text{or} \quad \mathbf{w} = \tfrac{1}{2}\nabla_{\mathbf{X}} \times \mathbf{u} \tag{3.50}$$

in terms of which the relative displacement is given by the expression

$$du_i = \epsilon_{ijk}w_j\,dX_k \quad \text{or} \quad d\mathbf{u} = \mathbf{w} \times d\mathbf{X} \tag{3.51}$$

The development of the Lagrangian description of the relative displacement vector, the linear rotation tensor and the linear rotation vector is paralleled completely by an analogous development for the Eulerian counterparts of these quantities. Accordingly the *Eulerian description* of the *relative displacement vector* is given by

$$du_i = \frac{\partial u_i}{\partial x_j}dx_j \quad \text{or} \quad d\mathbf{u} = \mathbf{K}\cdot d\mathbf{x} \tag{3.52}$$

and the *unit relative displacement vector* by

$$du_i = \frac{\partial u_i}{\partial x_j}\frac{dx_j}{dx} = \frac{\partial u_i}{\partial x_j}\mu_j \quad \text{or} \quad \frac{d\mathbf{u}}{dx} = \mathbf{u}\nabla_{\mathbf{x}}\cdot\hat{\boldsymbol{\mu}} = \mathbf{K}\cdot\hat{\boldsymbol{\mu}} \tag{3.53}$$

Decomposition of the Eulerian displacement gradient $\partial u_i/\partial x_j$ results in the expression

$$\frac{du_i}{dx} = \left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\right] dx_j$$

or

$$d\mathbf{u} = [\tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{x}} + \nabla_{\mathbf{x}}\mathbf{u}) + \tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{x}} - \nabla_{\mathbf{x}}\mathbf{u})] \cdot d\mathbf{x} \tag{3.54}$$

The first term in the square brackets of (*3.54*) is the Eulerian linear strain tensor ϵ_{ij}. The second term is the *linear Eulerian rotation tensor* and is denoted by

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \quad \text{or} \quad \mathbf{\Omega} = \tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{x}} - \nabla_{\mathbf{x}}\mathbf{u}) \tag{3.55}$$

From (*3.55*), the *linear Eulerian rotation vector* is defined by

$$\omega_i = \tfrac{1}{2}\epsilon_{ijk}\omega_{kj} \quad \text{or} \quad \boldsymbol{\omega} = \tfrac{1}{2}\nabla_{\mathbf{x}} \times \mathbf{u} \tag{3.56}$$

in terms of which the relative displacement is given by the expression

$$du_i = \epsilon_{ijk}\omega_j\,dx_k \quad \text{or} \quad d\mathbf{u} = \boldsymbol{\omega} \times d\mathbf{x} \tag{3.57}$$

3.9 INTERPRETATION OF THE LINEAR STRAIN TENSORS

For small deformation theory, the finite Lagrangian strain tensor L_{ij} in (*3.36*) may be replaced by the linear Lagrangian strain tensor l_{ij}, and that expression may now be written

$$(dx)^2 - (dX)^2 = (dx - dX)(dx + dX) = 2l_{ij}\, dX_i\, dX_j$$

or

$$(dx)^2 - (dX)^2 = (dx - dX)(dx + dX) = d\mathbf{X}\cdot 2\mathbf{L}\cdot d\mathbf{X} \qquad (3.58)$$

Since $dx \approx dX$ for small deformations, this equation may be put into the form

$$\frac{dx - dX}{dX} = l_{ij}\frac{dX_i}{dX}\frac{dX_j}{dX} = l_{ij}\nu_i\nu_j \quad \text{or} \quad \frac{dx - dX}{dX} = \hat{\boldsymbol{\nu}}\cdot\mathbf{L}\cdot\hat{\boldsymbol{\nu}} \qquad (3.59)$$

The left-hand side of (3.59) is recognized as the change in length per unit original length of the differential element and is called the *normal strain* for the line element originally having direction cosines dX_i/dX.

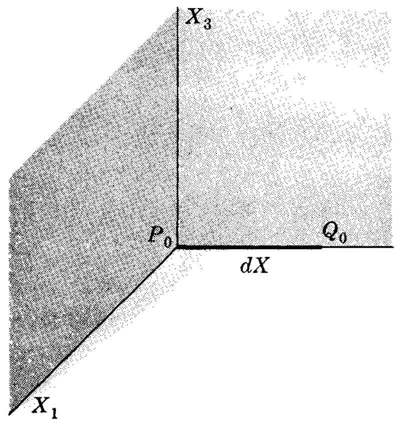


Fig. 3-4

When (3.59) is applied to the differential line element P_0Q_0, located with respect to the set of local axes at P_0 as shown in Fig. 3-4, the result will be the normal strain for that element. Because P_0Q_0 here lies along the X_2 axis,

$$dX_1/dX = dX_3/dX = 0, \quad dX_2/dX = 1$$

and therefore (3.59) becomes

$$\frac{dx - dX}{dX} = l_{22} = \frac{\partial u_2}{\partial X_2} \qquad (3.60)$$

Thus the normal strain for an element originally along the X_2 axis is seen to be the component l_{22}. Likewise for elements originally situated along the X_1 and X_3 axes, (3.59) yields normal strain values l_{11} and l_{33} respectively. In general, therefore, the diagonal terms of the linear strain tensor represent normal strains in the coordinate directions.

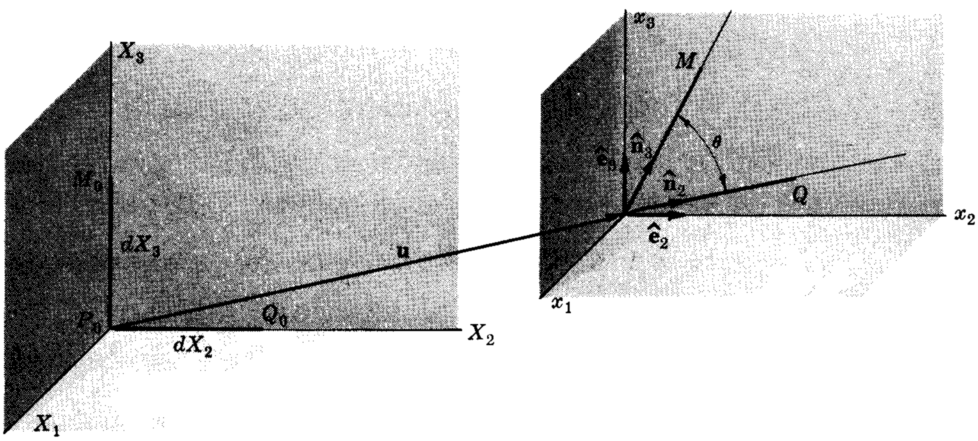


Fig. 3-5

The physical interpretation of the off-diagonal terms of l_{ij} may be obtained by a consideration of the line elements originally located along two of the coordinate axes. In Fig. 3-5 the line elements P_0Q_0 and P_0M_0 originally along the X_2 and X_3 axes, respectively, become after deformation the line elements PQ and PM with respect to the parallel set of local axes with origin at P. The original right angle between the line elements becomes the angle θ. From (3.46) and the assumption of small deformation theory, a first order approximation gives the unit vector at P in the direction of Q as

$$\hat{\mathbf{n}}_2 = \frac{\partial u_1}{\partial X_2}\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \frac{\partial u_3}{\partial X_2}\hat{\mathbf{e}}_3 \tag{3.61}$$

and, for the unit vector at P in the direction of M, as

$$\hat{\mathbf{n}}_3 = \frac{\partial u_1}{\partial X_3}\hat{\mathbf{e}}_1 + \frac{\partial u_2}{\partial X_3}\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 \tag{3.62}$$

Therefore
$$\cos\theta = \hat{\mathbf{n}}_2 \cdot \hat{\mathbf{n}}_3 = \frac{\partial u_1}{\partial X_3}\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} \tag{3.63}$$

or, neglecting the product term which is of higher order,

$$\cos\theta = \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} = 2l_{23} \tag{3.64}$$

Furthermore, taking the change in the right angle between the elements as $\gamma_{23} = \pi/2 - \theta$, and remembering that for the linear theory γ_{23} is very small, it follows that

$$\gamma_{23} \approx \sin\gamma_{23} = \sin(\pi/2 - \theta) = \cos\theta = 2l_{23} \tag{3.65}$$

Therefore the off-diagonal terms of the linear strain tensor represent one-half the angle change between two line elements originally at right angles to one another. These strain components are called *shearing strains,* and because of the factor 2 in (*3.65*) these tensor components are equal to one-half the familiar "engineering" shearing strains.

A development, essentially paralleling the one just presented for the interpretation of the components of l_{ij}, may also be made for the linear Eulerian strain tensor ϵ_{ij}. The essential difference in the derivations rests in the choice of line elements, which in the Eulerian description must be those that lie along the coordinate axes after deformation. The diagonal terms of ϵ_{ij} are the normal strains, and the off-diagonal terms the shearing strains. For those deformations in which the assumption $l_{ij} = \epsilon_{ij}$ is valid, no distinction is made between the Eulerian and Lagrangian interpretations.

3.10 STRETCH RATIO. FINITE STRAIN INTERPRETATION

An important measure of the extensional strain of a differential line element is the ratio dx/dX, known as the *stretch* or *stretch ratio.* This quantity may be defined at either the point P_0 in the undeformed configuration or at the point P in the deformed configuration. Thus from (*3.34*) the squared stretch at point P_0 for the line element along the unit vector $\hat{\mathbf{m}} = d\mathbf{X}/dX$, is given by

$$\left(\frac{dx}{dX}\right)^2_{P_0} = \Lambda^2_{(\hat{\mathbf{m}})} = G_{ij}\frac{dX_i}{dX}\frac{dX_j}{dX} \quad \text{or} \quad \Lambda^2_{(\hat{\mathbf{m}})} = \hat{\mathbf{m}}\cdot\mathbf{G}\cdot\hat{\mathbf{m}} \tag{3.66}$$

Similarly, from (*3.30*) the reciprocal of the squared stretch for the line element at P along the unit vector $\hat{\mathbf{n}} = d\mathbf{x}/dx$ is given by

$$\left(\frac{dX}{dx}\right)^2_{P} = \frac{1}{\lambda^2_{(\hat{\mathbf{n}})}} = C_{ij}\frac{dx_i}{dx}\frac{dx_j}{dx} \quad \text{or} \quad \frac{1}{\lambda^2_{(\hat{\mathbf{n}})}} = \hat{\mathbf{n}}\cdot\mathbf{C}\cdot\hat{\mathbf{n}} \tag{3.67}$$

For an element originally along the local X_2 axis shown in Fig. 3-4, $\hat{\mathbf{m}} \equiv \hat{\mathbf{e}}_2$ and therefore $dX_1/dX = dX_3/dX = 0$, $dX_2/dX = 1$ so that (*3.66*) yields for such an element

$$\Lambda^2_{(\hat{\mathbf{e}}_2)} = G_{22} = 1 + 2L_{22} \tag{3.68}$$

Similar results may be determined for $\Lambda^2_{(\hat{\mathbf{e}}_1)}$ and $\Lambda^2_{(\hat{\mathbf{e}}_3)}$.

For an element parallel to the x_2 axis after deformation, (*3.67*) yields the result

$$\frac{1}{\lambda^2_{(\hat{e}_2)}} = 1 - 2E_{22} \qquad (3.69)$$

with similar expressions for the quantities $1/\lambda^2_{(\hat{e}_1)}$ and $1/\lambda^2_{(\hat{e}_3)}$. In general, $\Lambda_{(\hat{e}_2)}$ is not equal to $\lambda_{(\hat{e}_2)}$ since the element originally along the X_2 axis will not likely lie along the x_2 axis after deformation.

The stretch ratio provides a basis for interpretation of the finite strain tensors. Thus the change of length per unit of original length is

$$\frac{dx - dX}{dX} = \frac{dx}{dX} - 1 = \Lambda_{(\hat{m})} - 1 \qquad (3.70)$$

and for the element P_0Q_0 along the X_2 axis (of Fig. 3-4), the *unit extension* is therefore

$$L_{(2)} = \Lambda_{(\hat{e}_2)} - 1 = \sqrt{1 + 2L_{22}} - 1 \qquad (3.71)$$

This result may also be derived directly from (*3.36*). For small deformation theory, (*3.71*) reduces to (*3.60*). Also, the unit extensions $L_{(1)}$ and $L_{(3)}$ are given by analogous equations in terms of L_{11} and L_{33} respectively.

For the two differential line elements shown in Fig. 3-5, the change in angle $\gamma_{23} = \pi/2 - \theta$ is given in terms of $\Lambda_{(\hat{e}_2)}$ and $\Lambda_{(\hat{e}_3)}$ by

$$\sin \gamma_{23} = \frac{2L_{23}}{\Lambda_{(\hat{e}_2)}\Lambda_{(\hat{e}_3)}} = \frac{2L_{23}}{\sqrt{1+2L_{22}}\sqrt{1+2L_{33}}} \qquad (3.72)$$

When deformations are small, (*3.72*) reduces to (*3.65*).

3.11 STRETCH TENSORS. ROTATION TENSOR

The so-called *polar decomposition* of an arbitrary, nonsingular, second-order tensor is given by the product of a positive symmetric second-order tensor with an orthogonal second-order tensor. When such a multiplicative decomposition is applied to the deformation gradient **F**, the result may be written

$$F_{ij} \equiv \frac{\partial x_i}{\partial X_j} = R_{ik}S_{kj} = T_{ik}R_{kj} \quad \text{or} \quad \mathbf{F} = \mathbf{R}\cdot\mathbf{S} = \mathbf{T}\cdot\mathbf{R} \qquad (3.73)$$

in which **R** is the *orthogonal rotation tensor,* and **S** and **T** are positive symmetric tensors known as the *right stretch tensor* and *left stretch tensor* respectively.

The interpretation of (*3.73*) is provided through the relationship $dx_i = (\partial x_i/\partial X_j)\, dX_j$ given by (*3.33*). Inserting the inner products of (*3.73*) into (*3.33*) results in the equations

$$dx_i = R_{ik}S_{kj}\,dX_j = T_{ik}R_{kj}\,dX_j \quad \text{or} \quad d\mathbf{x} = \mathbf{R}\cdot\mathbf{S}\cdot d\mathbf{X} = \mathbf{T}\cdot\mathbf{R}\cdot d\mathbf{X} \qquad (3.74)$$

From these expressions the deformation of dX_i into dx_i as illustrated in Fig. 3-2 may be given either of two physical interpretations. In the first form of the right hand side of (*3.74*), the deformation consists of a sequential stretching (by **S**) and rotation to be followed by a rigid body displacement to the point P. In the second form, a rigid body translation to P is followed by a rotation and finally the stretching (by **T**). The translation, of course, does not alter the vector components relative to the axes X_i and x_i.

3.12 TRANSFORMATION PROPERTIES OF STRAIN TENSORS

The various strain tensors L_{ij}, E_{ij}, l_{ij} and ϵ_{ij} defined respectively by (*3.37*), (*3.39*), (*3.42*) and (*3.43*) are all second-order Cartesian tensors as indicated by the two free indices in each. Accordingly for a set of rotated axes X_i' having the transformation matrix $[b_{ij}]$ with respect to the set of local unprimed axes X_i at point P_0 as shown in Fig. 3-6(*a*), the components of L_{ij}' and l_{ij}' are given by

$$L_{ij}' = b_{ip}b_{jq}L_{pq} \quad \text{or} \quad \mathbf{L}_G' = \mathbf{B}\cdot\mathbf{L}_G\cdot\mathbf{B}_c \qquad (3.75)$$

and

$$l_{ij}' = b_{ip}b_{jq}l_{pq} \quad \text{or} \quad \mathbf{L}' = \mathbf{B}\cdot\mathbf{L}\cdot\mathbf{B}_c \qquad (3.76)$$

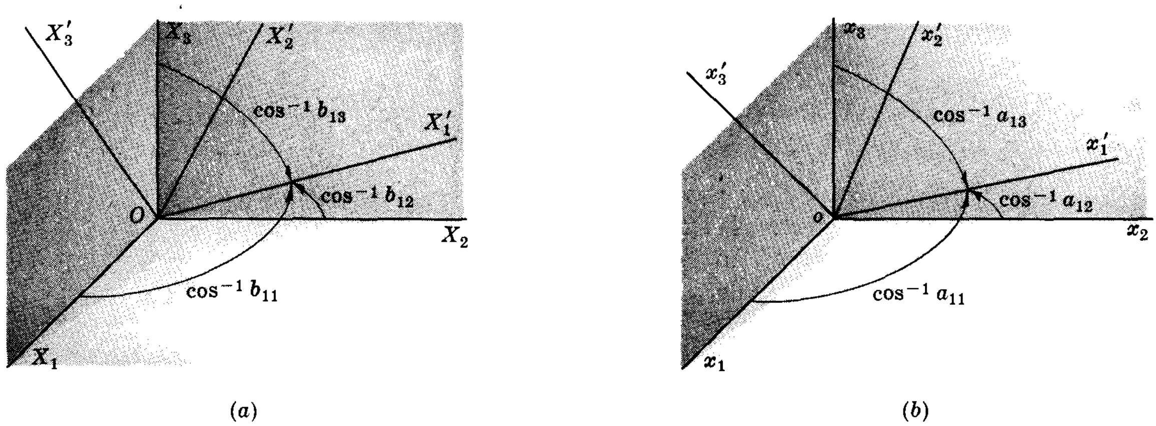


Fig. 3-6

Likewise, for the rotated axes x_i' having the transformation matrix $[a_{ij}]$ in Fig. 3-6(*b*), the components of E_{ij}' and e_{ij}' are given by

$$E_{ij}' = a_{ip}a_{jq}E_{pq} \quad \text{or} \quad \mathbf{E}_A' = \mathbf{A}\cdot\mathbf{E}_A\cdot\mathbf{A}_c \qquad (3.77)$$

and

$$\epsilon_{ij}' = a_{ip}a_{jq}\epsilon_{pq} \quad \text{or} \quad \mathbf{E}' = \mathbf{A}\cdot\mathbf{E}\cdot\mathbf{A}_c \qquad (3.78)$$

By analogy with the stress quadric described in Section 2.9, page 50, the *Lagrangian* and *Eulerian linear strain quadrics* may be given with reference to local Cartesian coordinates η_i and ζ_i at the points P_0 and P respectively as shown in Fig. 3-7. Thus the equation of the *Lagrangian strain quadric* is given by

$$l_{ij}\eta_i\eta_j = \pm h^2 \quad \text{or} \quad \boldsymbol{\eta}\cdot\mathbf{L}\cdot\boldsymbol{\eta} = \pm h^2 \qquad (3.79)$$

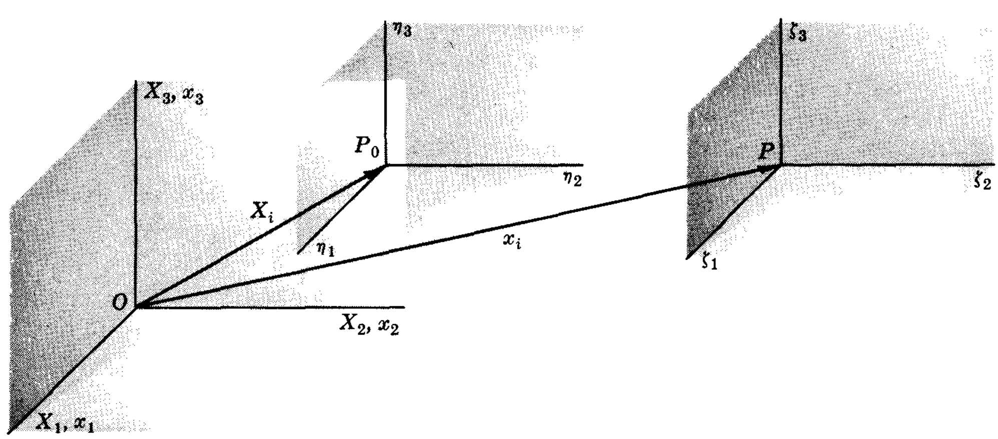


Fig. 3-7

and the equation of the *Eulerian strain quadric* is given by

$$\epsilon_{ij}\zeta_i\zeta_j = \pm g^2 \quad \text{or} \quad \boldsymbol{\zeta}\cdot\mathbf{E}\cdot\boldsymbol{\zeta} = \pm g^2 \tag{3.80}$$

Two important properties of the Lagrangian {Eulerian} linear strain quadric are:

1. The normal strain with respect to the original {final} length of a line element is inversely proportional to the distance squared from the origin of the quadric P_0 $\{P\}$ to a point on its surface.
2. The relative displacement of the neighboring particle located at Q_0 $\{Q\}$ per unit original {final} length is parallel to the normal of the quadric surface at the point of intersection with the line through P_0Q_0 $\{PQ\}$.

Additional insight into the nature of local deformations in the neighborhood of P_0 is provided by defining the *strain ellipsoid* at that point. Thus for the undeformed continuum, the equation of the bounding surface of an infinitesimal sphere of radius R is given in terms of local material coordinates by (*3.28*) as

$$(dX)^2 = \delta_{ij}\,dX_i\,dX_j = R^2 \quad \text{or} \quad (dX)^2 = d\mathbf{X}\cdot\mathbf{I}\cdot d\mathbf{X} = R^2 \tag{3.81}$$

After deformation, the equation of the surface of the same material particles is given by (*3.30*) as

$$(dX)^2 = C_{ij}\,dx_i\,dx_j = R^2 \quad \text{or} \quad (dX)^2 = d\mathbf{x}\cdot\mathbf{C}\cdot d\mathbf{x} = R^2 \tag{3.82}$$

which describes an ellipsoid, known as the *material strain ellipsoid.* Therefore a spherical volume of the continuum in the undeformed state is changed into an ellipsoid at P_0 by the deformation. By comparison, an infinitesimal spherical volume at P in the deformed continuum began as an ellipsoidal volume element in the undeformed state. For a sphere of radius r at P, the equations for these surfaces in terms of local coordinates are given by (*3.32*) for the sphere as

$$(dx)^2 = \delta_{ij}\,dx_i\,dx_j = r^2 \quad \text{or} \quad (dx)^2 = d\mathbf{x}\cdot\mathbf{I}\cdot d\mathbf{x} = r^2 \tag{3.83}$$

and by (*3.34*) for the ellipsoid as

$$(dx)^2 = G_{ij}\,dX_i\,dX_j = r^2 \quad \text{or} \quad (dx)^2 = d\mathbf{X}\cdot\mathbf{G}\cdot d\mathbf{X} = r^2 \tag{3.84}$$

The ellipsoid of (*3.84*) is called the *spatial strain ellipsoid.* Such strain ellipsoids as described here are frequently known as *Cauchy strain ellipsoids.*

3.13 PRINCIPAL STRAINS. STRAIN INVARIANTS. CUBICAL DILATATION

The Lagrangian and Eulerian linear strain tensors are symmetric second-order Cartesian tensors, and accordingly the determination of their principal directions and principal strain values follows the standard development presented in Section 1.19, page 20. Physically, a principal direction of the strain tensor is one for which the orientation of an element at a given point is not altered by a pure strain deformation. The principal strain value is simply the unit relative displacement (normal strain) that occurs in the principal direction.

For the Lagrangian strain tensor l_{ij}, the unit relative displacement vector is given by (*3.47*), which may be written

$$\frac{du_i}{dX} = (l_{ij} + W_{ij})\nu_j \quad \text{or} \quad \frac{d\mathbf{u}}{dX} = (\mathbf{L}+\mathbf{W})\cdot\hat{\boldsymbol{\nu}} \tag{3.85}$$

Calling $l_i^{(\hat{n})}$ the normal strain in the direction of the unit vector n_i, (*3.85*) yields for pure strain ($W_{ij} \equiv 0$) the relation

$$l_i^{(\hat{n})} = l_{ij}n_j \quad \text{or} \quad \mathbf{l}^{(\hat{n})} = \mathbf{L}\cdot\hat{\mathbf{n}} \tag{3.86}$$

If the direction n_i is a principal direction with a principal strain value l, then

$$l_i^{(\hat{n})} = ln_i = l\delta_{ij}n_j \quad \text{or} \quad \mathbf{l}^{(\hat{n})} = l\hat{\mathbf{n}} = l\mathbf{I}\cdot\hat{\mathbf{n}} \tag{3.87}$$

Equating the right-hand sides of (3.86) and (3.87) leads to the relationship

$$(l_{ij} - \delta_{ij}l)n_j = 0 \quad \text{or} \quad (\mathbf{L} - \mathbf{I}l)\cdot\hat{\mathbf{n}} = 0 \tag{3.88}$$

which together with the condition $n_in_i = 1$ on the unit vectors n_i provide the necessary equations for determining the principal strain value l and its direction cosines n_i. Nontrivial solutions of (3.88) exist if and only if the determinant of coefficients vanishes. Therefore

$$|l_{ij} - \delta_{ij}l| = 0 \quad \text{or} \quad |\mathbf{L} - \mathbf{I}l| = 0 \tag{3.89}$$

which upon expansion yields the characteristic equation of l_{ij}, the cubic

$$l^3 - \mathrm{I}_L l^2 + \mathrm{II}_L l - \mathrm{III}_L = 0 \tag{3.90}$$

where $\quad \mathrm{I}_L = l_{ii} = \operatorname{tr}\mathbf{L}, \quad \mathrm{II}_L = \frac{1}{2}(l_{ii}l_{jj} - l_{ij}l_{ij}), \quad \mathrm{III}_L = |l_{ij}| = \det\mathbf{L} \qquad (3.91)$

are the first, second and third *Lagrangian strain invariants* respectively. The roots of (3.90) are the principal strain values denoted by $l_{(1)}$, $l_{(2)}$ and $l_{(3)}$.

The first invariant of the Lagrangian strain tensor may be expressed in terms of the principal strains as

$$\mathrm{I}_L = l_{ii} = l_{(1)} + l_{(2)} + l_{(3)} \tag{3.92}$$

and has an important physical interpretation. To see this, consider a differential rectangular parallelepiped whose edges are parallel to the principal strain directions as shown in Fig. 3-8. The change in volume per unit original volume of this element is called the *cubical dilatation* and is given by

$$D_0 = \frac{\Delta V_0}{V_0} = \frac{dX_1(1 + l_{(1)})\,dX_2(1 + l_{(2)})\,dX_3(1 + l_{(3)}) - dX_1\,dX_2\,dX_3}{dX_1\,dX_2\,dX_3} \tag{3.93}$$

For small strain theory, the first-order approximation of this ratio is the sum

$$D_0 = l_{(1)} + l_{(2)} + l_{(3)} = \mathrm{I}_L \tag{3.94}$$

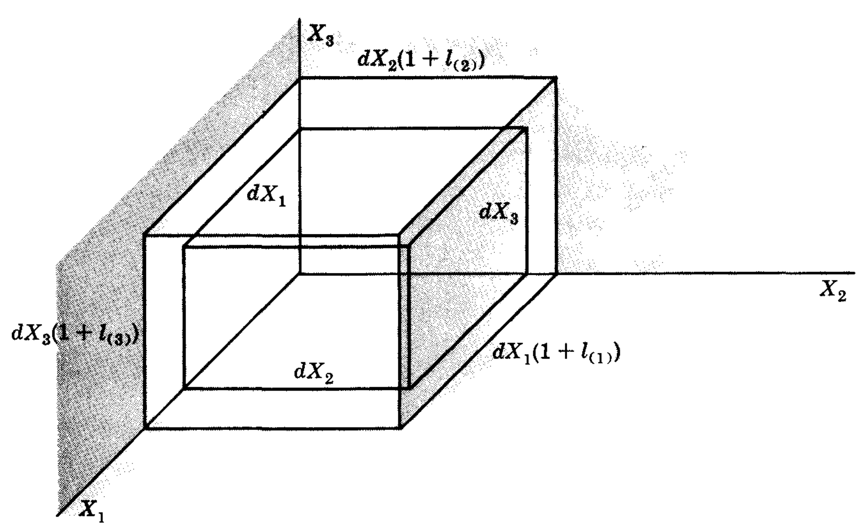


Fig. 3-8

With regard to the Eulerian strain tensor ϵ_{ij} and its associated unit relative displacement vector $\epsilon_i^{(\hat{n})}$, the principal directions and principal strain values $\epsilon_{(1)}, \epsilon_{(2)}, \epsilon_{(3)}$ are determined in exactly the same way as their Lagrangian counterparts. The Eulerian strain invariants may be expressed in terms of the principal strains as

$$\begin{aligned} \mathrm{I}_E &= \epsilon_{(1)} + \epsilon_{(2)} + \epsilon_{(3)} \\ \mathrm{II}_E &= \epsilon_{(1)}\epsilon_{(2)} + \epsilon_{(2)}\epsilon_{(3)} + \epsilon_{(3)}\epsilon_{(1)} \\ \mathrm{III}_E &= \epsilon_{(1)}\epsilon_{(2)}\epsilon_{(3)} \end{aligned} \tag{3.95}$$

The cubical dilatation for the Eulerian description is given by

$$\Delta V/V = D = \epsilon_{(1)} + \epsilon_{(2)} + \epsilon_{(3)} \tag{3.96}$$

3.14 SPHERICAL AND DEVIATOR STRAIN TENSORS

The Lagrangian and Eulerian *linear* strain tensors may each be split into a *spherical* and *deviator* tensor in the same manner in which the stress tensor decomposition was carried out in Chapter 2. As before, if Lagrangian and Eulerian deviator tensor components are denoted by d_{ij} and e_{ij} respectively, the resolution expressions are

$$l_{ij} = d_{ij} + \delta_{ij}\frac{l_{kk}}{3} \quad \text{or} \quad \mathbf{L} = \mathbf{L}_D + \frac{\mathbf{I}(\text{tr}\,\mathbf{L})}{3} \tag{3.97}$$

and

$$\epsilon_{ij} = e_{ij} + \delta_{ij}\frac{\epsilon_{kk}}{3} \quad \text{or} \quad \mathbf{E} = \mathbf{E}_D + \frac{\mathbf{I}(\text{tr}\,\mathbf{E})}{3} \tag{3.98}$$

The deviator tensors are associated with shear deformation for which the cubical dilatation vanishes. Therefore it is not surprising that the first invariants d_{ii} and e_{ii} of the deviator strain tensors are identically zero.

3.15 PLANE STRAIN. MOHR'S CIRCLES FOR STRAIN

When one and only one of the principal strains at a point in a continuum is zero, a state of *plane strain* is said to exist at that point. In the Eulerian description (the Lagrangian description follows exactly the same pattern), if x_3 is taken as the direction of the zero principal strain, a state of plane strain parallel to the x_1x_2 plane exists and the linear strain tensor is given by

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad [\epsilon_{ij}] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.99}$$

When x_1 and x_2 are also principal directions, the strain tensor has the form

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{(1)} & 0 & 0 \\ 0 & \epsilon_{(2)} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad [\epsilon_{ij}] = \begin{bmatrix} \epsilon_{(1)} & 0 & 0 \\ 0 & \epsilon_{(2)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.100}$$

In many books on "Strength of Materials" and "Elasticity", plane strain is referred to as *plane deformation* since the deformation field is identical in all planes perpendicular to the direction of the zero principal strain. For plane strain perpendicular to the x_3 axis, the displacement vector may be taken as a function of x_1 and x_2 only. The appropriate displacement components for this case of plane strain are designated by

$$\begin{aligned} u_1 &= u_1(x_1, x_2) \\ u_2 &= u_2(x_1, x_2) \\ u_3 &= C \text{ (a constant, usually taken as zero)} \end{aligned} \qquad (3.101)$$

Inserting these expressions into the definition of ϵ_{ij} given by (*3.43*) produces the plane strain tensor in the same form shown in (*3.99*).

A graphical description of the state of strain at a point is provided by the *Mohr's circles for strain* in a manner exactly like that presented in Chapter 2 for the Mohr's circles for stress. For this purpose the strain tensor is often displayed in the form

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{11} & \frac{1}{2}\gamma_{12} & \frac{1}{2}\gamma_{13} \\ \frac{1}{2}\gamma_{12} & \epsilon_{22} & \frac{1}{2}\gamma_{23} \\ \frac{1}{2}\gamma_{13} & \frac{1}{2}\gamma_{23} & \epsilon_{33} \end{pmatrix} \qquad (3.102)$$

Here the γ_{ij} (with $i \neq j$) are the so-called "engineering" shear strain components, which are twice the tensorial shear strain components.

The state of strain at an unloaded point on the bounding surface of a continuum body is locally plane strain. Frequently in experimental studies involving strain measurements at such a surface point, Mohr's strain circles are useful for reporting the observed data. Usually three normal strains are measured at the given point by means of a strain rosette, and the Mohr's circles diagram constructed from these. Corresponding to the plane stress Mohr's circles, a typical case of plane strain diagram is shown in Fig. 3-9. The principal normal strains are labeled as such in the diagram, and the maximum shear strain values are represented by points D and E.

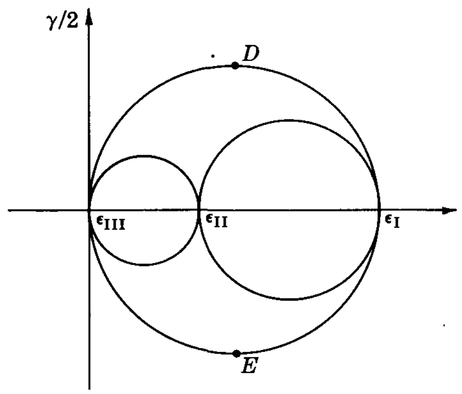


Fig. 3-9

3.16 COMPATIBILITY EQUATIONS FOR LINEAR STRAINS

If the strain components ϵ_{ij} are given explicitly as functions of the coordinates, the six independent equations (*3.43*)

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

may be viewed as a system of six partial differential equations for determining the three displacement components u_i. The system is over-determined and will not, in general, possess a solution for an arbitrary choice of the strain components ϵ_{ij}. Therefore if the displacement components u_i are to be single-valued and continuous, some conditions must be imposed upon the strain components. The necessary and sufficient conditions for such a displacement field are expressed by the equations

$$\frac{\partial^2 \epsilon_{ij}}{\partial x_k \, \partial x_m} + \frac{\partial^2 \epsilon_{km}}{\partial x_i \, \partial x_j} - \frac{\partial^2 \epsilon_{ik}}{\partial x_j \, \partial x_m} - \frac{\partial^2 \epsilon_{jm}}{\partial x_i \, \partial x_k} = 0 \qquad (3.103)$$

There are eighty-one equations in all in (*3.103*) but only six are distinct. These six written in explicit and symbolic form appear as

$$\left.\begin{array}{lll}
1. & \dfrac{\partial^2\epsilon_{11}}{\partial x_2^2} + \dfrac{\partial^2\epsilon_{22}}{\partial x_1^2} & = 2\dfrac{\partial^2\epsilon_{12}}{\partial x_1\,\partial x_2} \\
2. & \dfrac{\partial^2\epsilon_{22}}{\partial x_3^2} + \dfrac{\partial^2\epsilon_{33}}{\partial x_2^2} & = 2\dfrac{\partial^2\epsilon_{23}}{\partial x_2\,\partial x_3} \\
3. & \dfrac{\partial^2\epsilon_{33}}{\partial x_1^2} + \dfrac{\partial^2\epsilon_{11}}{\partial x_3^2} & = 2\dfrac{\partial^2\epsilon_{31}}{\partial x_3\,\partial x_1} \\
4. & \dfrac{\partial}{\partial x_1}\left(-\dfrac{\partial\epsilon_{23}}{\partial x_1} + \dfrac{\partial\epsilon_{31}}{\partial x_2} + \dfrac{\partial\epsilon_{12}}{\partial x_3}\right) & = \dfrac{\partial^2\epsilon_{11}}{\partial x_2\,\partial x_3} \\
5. & \dfrac{\partial}{\partial x_2}\left(\dfrac{\partial\epsilon_{23}}{\partial x_1} - \dfrac{\partial\epsilon_{31}}{\partial x_2} + \dfrac{\partial\epsilon_{12}}{\partial x_3}\right) & = \dfrac{\partial^2\epsilon_{22}}{\partial x_3\,\partial x_1} \\
6. & \dfrac{\partial}{\partial x_3}\left(\dfrac{\partial\epsilon_{23}}{\partial x_1} + \dfrac{\partial\epsilon_{31}}{\partial x_2} - \dfrac{\partial\epsilon_{12}}{\partial x_3}\right) & = \dfrac{\partial^2\epsilon_{33}}{\partial x_1\,\partial x_2}
\end{array}\right\} \quad \text{or} \quad \nabla_{\mathbf{x}} \times \mathbf{E} \times \nabla_{\mathbf{x}} = 0 \qquad (3.104)$$

Compatibility equations in terms of the Lagrangian linear strain tensor l_{ij} may also be written down by an obvious correspondence to the Eulerian form given above. For plane strain parallel to the x_1x_2 plane, the six equations in (*3.104*) reduce to the single equation

$$\frac{\partial^2\epsilon_{11}}{\partial x_2^2} + \frac{\partial^2\epsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2\epsilon_{12}}{\partial x_1\,\partial x_2} \quad \text{or} \quad \nabla_{\mathbf{x}} \times \mathbf{E} \times \nabla_{\mathbf{x}} = 0 \qquad (3.105)$$

where $\mathbf{E}$ is of the form given by (*3.99*).

Solved Problems

DISPLACEMENT AND DEFORMATION (Sec. 3.1-3.5)

3.1. With respect to superposed material axes X_i and spatial axes x_i, the displacement field of a continuum body is given by $x_1 = X_1$, $x_2 = X_2 + AX_3$, $x_3 = X_3 + AX_2$ where A is a constant. Determine the displacement vector components in both the material and spatial forms.

From (*3.13*) directly, the displacement components in material form are $u_1 = x_1 - X_1 = 0$, $u_2 = x_2 - X_2 = AX_3$, $u_3 = x_3 - X_3 = AX_2$. Inverting the given displacement relations to obtain $X_1 = x_1$, $X_2 = (x_2 - Ax_3)/(1 - A^2)$, $X_3 = (x_3 - Ax_2)/(1 - A^2)$, the spatial components of $\mathbf{u}$ are $u_1 = 0$, $u_2 = A(x_3 - Ax_2)/(1 - A^2)$, $u_3 = A(x_2 - Ax_3)/(1 - A^2)$.

From these results it is noted that the originally straight line of material particles expressed by $X_1 = 0$, $X_2 + X_3 = 1/(1 + A)$ occupies the location $x_1 = 0$, $x_2 + x_3 = 1$ after displacement. Likewise the particle line $X_1 = 0$, $X_2 = X_3$ becomes after displacement $x_1 = 0$, $x_2 = x_3$. (Interpret the physical meaning of this.)

3.2. For the displacement field of Problem 3.1 determine the displaced location of the material particles which originally comprise (*a*) the plane circular surface $X_1 = 0$, $X_2^2 + X_3^2 = 1/(1 - A^2)$, (*b*) the infinitesimal cube with edges along the coordinate axes of length $dX_i = dX$. Sketch the displaced configurations for (*a*) and (*b*) if $A = \frac{1}{2}$.

(*a*) By the direct substitutions $X_2 = (x_2 - Ax_3)/(1 - A^2)$ and $X_3 = (x_3 - Ax_2)/(1 - A^2)$, the circular surface becomes the elliptical surface $(1 + A^2)x_2^2 - 4Ax_2x_3 + (1 + A^2)x_3^2 = (1 - A^2)$. For $A = \frac{1}{2}$, this is bounded by the ellipse $5x_2^2 - 8x_2x_3 + 5x_3^2 = 3$ which when referred to its principal axes x_i^* (at 45° with x_i, $i = 2, 3$) has the equation $x_2^{*2} + 9x_3^{*2} = 3$. Fig. 3-10 below shows this displacement pattern.

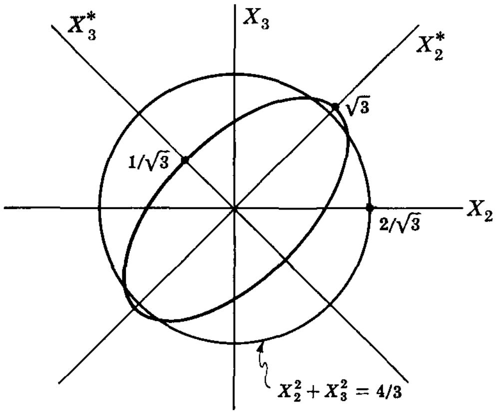


Fig. 3-10

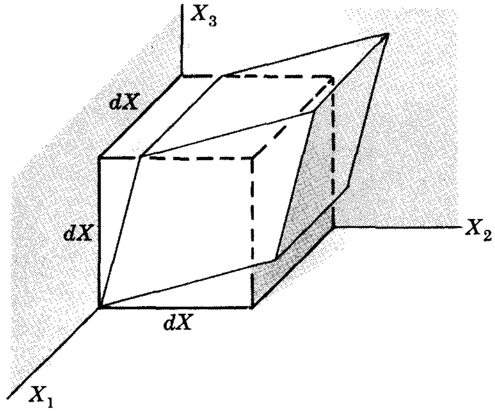


Fig. 3-11

(*b*) From Problem 3.1, the displacements of the edges of the cube are readily calculated. For the edge $X_1 = X_1$, $X_2 = X_3 = 0$, $u_1 = u_2 = u_3 = 0$. For the edge $X_1 = 0 = X_2$, $X_3 = X_3$, $u_1 = u_3 = 0$, $u_2 = AX_3$ and the particles on this edge are displaced in the X_2 direction proportionally to their distance from the origin. For the edge $X_1 = X_3 = 0$, $X_2 = X_2$, $u_1 = u_2 = 0$, $u_3 = AX_2$. The initial and displaced positions of the cube are shown in Fig. 3-11.

3.3. For superposed material and spatial axes, the displacement vector of a body is given by $\mathbf{u} = 4X_1^2\hat{\mathbf{e}}_1 + X_2X_3^2\hat{\mathbf{e}}_2 + X_1X_3^2\hat{\mathbf{e}}_3$. Determine the displaced location of the particle originally at $(1, 0, 2)$.

The original position vector of the particle is $\mathbf{X} = \hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_3$. Its displacement is $\mathbf{u} = 4\,\hat{\mathbf{e}}_1 + 4\,\hat{\mathbf{e}}_3$ and since $\mathbf{x} = \mathbf{X} + \mathbf{u}$, its final position vector is $\mathbf{x} = 5\,\hat{\mathbf{e}}_1 + 6\,\hat{\mathbf{e}}_3$.

3.4. With respect to rectangular Cartesian material coordinates X_i, a displacement field is given by $U_1 = -AX_2X_3$, $U_2 = AX_1X_3$, $U_3 = 0$ where A is a constant. Determine the displacement components for cylindrical spatial coordinates x_i if the two systems have a common origin.

From the geometry of the axes (Fig. 3-12) the transformation tensor $\alpha_{pK} = \hat{\mathbf{e}}_p \cdot \hat{\mathbf{I}}_K$ is

$$\alpha_{pK} = \begin{pmatrix} \cos x_2 & \sin x_2 & 0 \\ -\sin x_2 & \cos x_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

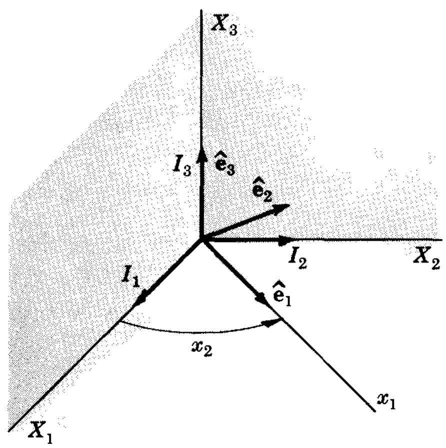


Fig. 3-12

and from the inverse form of (*3.9*) $u_p = \alpha_{pK}U_K$. Thus since Cartesian and cylindrical coordinates are related through the equations $X_1 = x_1 \cos x_2$, $X_2 = x_1 \sin x_2$, $X_3 = x_3$, equation (*3.9*) gives

$$\begin{aligned} u_1 &= (-\cos x_2)AX_2X_3 + (\sin x_2)AX_1X_3 \\ &= (-\cos x_2)Ax_3x_1 \sin x_2 + (\sin x_2)Ax_3x_1 \cos x_2 \;=\; 0 \\ u_2 &= (\sin x_2)AX_2X_3 + (\cos x_2)AX_1X_3 \\ &= (\sin^2 x_2)Ax_1x_3 + (\cos^2 x_2)Ax_1x_3 \;=\; Ax_1x_3 \\ u_3 &= 0 \end{aligned}$$

This displacement is that of a circular shaft in torsion.

3.5. The Lagrangian description of a deformation is given by $x_1 = X_1 + X_3(e^2 - 1)$, $x_2 = X_2 + X_3(e^2 - e^{-2})$, $x_3 = e^2 X_3$ where e is a constant. Show that the Jacobian J does not vanish and determine the Eulerian equations describing this motion.

$$\text{From } (3.16), \quad J = \begin{vmatrix} 1 & 0 & (e^2-1) \\ 0 & 1 & (e^2 - e^{-2}) \\ 0 & 0 & (e^2) \end{vmatrix} = e^2 \neq 0.$$

Inverting the equations, $X_1 = x_1 + x_3(e^{-2} - 1)$, $X_2 = x_2 + x_3(e^{-4} - 1)$, $X_3 = e^{-2}x_3$.

3.6. A displacement field is given by $\mathbf{u} = X_1X_3^2\,\hat{\mathbf{e}}_1 + X_1^2X_2\hat{\mathbf{e}}_2 + X_2^2X_3\hat{\mathbf{e}}_3$. Determine independently the material deformation gradient F and the material displacement gradient J and verify (*3.24*), $\mathsf{J} = \mathsf{F} - \mathsf{I}$.

From the given displacement vector $\mathbf{u}$, J is found to be

$$\frac{\partial u_i}{\partial X_j} = \begin{pmatrix} X_3^2 & 0 & 2X_1X_3 \\ 2X_1X_2 & X_1^2 & 0 \\ 0 & 2X_2X_3 & X_2^2 \end{pmatrix}$$

Since $\mathbf{x} = \mathbf{u} + \mathbf{X}$, the displacement field may also be described by equations $x_1 = X_1(1 + X_3^2)$, $x_2 = X_2(1 + X_1^2)$, $x_3 = X_3(1 + X_2^2)$ from which F is readily found to be

$$\partial x_i/\partial X_j = \begin{pmatrix} 1 + X_3^2 & 0 & 2X_1X_3 \\ 2X_1X_2 & 1 + X_1^2 & 0 \\ 0 & 2X_2X_3 & 1 + X_2^2 \end{pmatrix}$$

Direct substitution of the calculated tensors F and J into (*3.24*) verifies that the equation is satisfied.

3.7. A continuum body undergoes the displacement $\mathbf{u} = (3X_2 - 4X_3)\hat{\mathbf{e}}_1 + (2X_1 - X_3)\hat{\mathbf{e}}_2 + (4X_2 - X_1)\hat{\mathbf{e}}_3$. Determine the displaced position of the vector joining particles $A(1,0,3)$ and $B(3,6,6)$, assuming superposed material and spatial axes.

From (*3.13*), the spatial coordinates for this displacement are $x_1 = X_1 + 3X_2 - 4X_3$, $x_2 = 2X_1 + X_2 - X_3$, $x_3 = -X_1 + 4X_2 + X_3$. Thus the displaced position of particle A is given by $x_1 = -11$, $x_2 = -1$, $x_3 = 2$; and of particle B, $x_1 = -3$, $x_2 = 6$, $x_3 = 27$. Therefore the displaced position of the vector joining A and B may be written $\mathbf{V} = 8\,\hat{\mathbf{e}}_1 + 7\,\hat{\mathbf{e}}_2 + 25\,\hat{\mathbf{e}}_3$.

3.8. For the displacement field of Problem 3.7 determine the displaced position of the position vector of particle $C(2,6,3)$ which is parallel to the vector joining particles A and B. Show that the two vectors remain parallel after deformation.

By the analysis of Problem 3.7 the position vector of C becomes $\mathbf{U} = 8\,\hat{\mathbf{e}}_1 + 7\,\hat{\mathbf{e}}_2 + 25\,\hat{\mathbf{e}}_3$ which is clearly parallel to $\mathbf{V}$. This is an example of so-called *homogeneous deformation.*

3.9. The general formulation of homogeneous deformation is given by the displacement field $u_i = A_{ij}X_j$ where the A_{ij} are constants or at most functions of time. Show that this deformation is such that (*a*) plane sections remain plane, (*b*) straight lines remain straight.

(*a*) From (*3.13*), $\quad x_i = X_i + u_i = X_i + A_{ij}X_j = (\delta_{ij} + A_{ij})X_j$

According to (3.16) the inverse equations $X_i = (\delta_{ij} + B_{ij})x_j$ exist provided the determinant $|\delta_{ij} + A_{ij}|$ does not vanish. Assuming this is the case, the material plane $\beta_i X_i + \alpha = 0$ becomes $\beta_i(\delta_{ij} + B_{ij})x_j + \alpha = 0$ which may be written in standard form as the plane $\lambda_j x_j + \alpha = 0$ where the coefficients $\lambda_j = \beta_i(\delta_{ij} + B_{ij})$.

(b) A straight line may be considered as the intersection of two planes. In the deformed geometry, planes remain plane as proven and hence the intersection of two planes remains a straight line.

3.10. An *infinitesimal* homogeneous deformation $u_i = A_{ij}X_j$ is one for which the coefficients A_{ij} are so small that their products may be neglected in comparison to the coefficients themselves. Show that the total deformation resulting from two successive infinitesimal homogeneous deformations may be considered as the sum of the two individual deformations, and that the order of applying the displacements does not alter the final configuration.

Let $x_i = (\delta_{ij} + A_{ij})X_j$ and $x_i' = (\delta_{ij} + B_{ij})x_j$ be successive infinitesimal homogeneous displacements. Then $x_i' = (\delta_{ij} + B_{ij})(\delta_{jk} + A_{jk})X_k = (\delta_{ik} + B_{ik} + A_{ik} + B_{ij}A_{jk})X_k$. Neglecting the higher order product terms $B_{ij}A_{jk}$, this becomes $x_i' = (\delta_{ik} + B_{ik} + A_{ik})X_k = (\delta_{ik} + C_{ik})X_k$ which represents the infinitesimal homogeneous deformation

$$u_i'' = x_i' - X_i = C_{ik}X_k = (B_{ik} + A_{ik})X_k = (A_{ik} + B_{ik})X_k = u_i + u_i'$$

DEFORMATION AND STRAIN TENSORS (Sec. 3.6-3.9)

3.11. A continuum body undergoes the deformation $x_1 = X_1$, $x_2 = X_2 + AX_3$, $x_3 = X_3 + AX_2$ where A is a constant. Compute the deformation tensor $\mathbf{G}$ and use this to determine the Lagrangian finite strain tensor $\mathbf{L}_G$.

From (3.35), $\mathbf{G} = \mathbf{F}_c \cdot \mathbf{F}$ and by (3.20) $\mathbf{F}$ is given in matrix form as

$$[\partial x_i/\partial X_j] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & A \\ 0 & A & 1 \end{bmatrix} \quad \text{so that} \quad [G_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+A^2 & 2A \\ 0 & 2A & 1+A^2 \end{bmatrix}$$

Therefore from (3.37),

$$\mathbf{L}_G = \tfrac{1}{2}(\mathbf{G} - \mathbf{I}) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & A^2 & 2A \\ 0 & 2A & A^2 \end{pmatrix}$$

3.12. For the displacement field of Problem 3.11 calculate the squared length $(dx)^2$ of the edges OA and OB, and the diagonal OC after deformation for the small rectangle shown in Fig. 3-13.

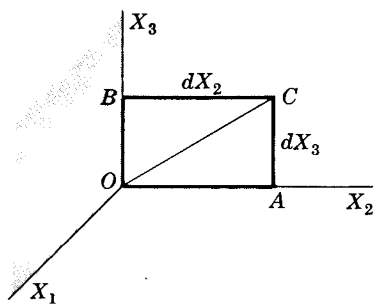


Fig. 3-13

Using $\mathbf{G}$ as determined in Problem 3.11 in (3.34), the squared length of the diagonal OC is given in matrix form by

$$(dx)^2 = [0, dX_2, dX_3]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+A^2 & 2A \\ 0 & 2A & 1+A^2 \end{bmatrix}\begin{bmatrix} 0 \\ dX_2 \\ dX_3 \end{bmatrix}$$

$$= (1+A^2)(dX_2)^2 + 4A\,dX_2\,dX_3 + (1+A^2)(dX_3)^2$$

Similarly for OA, $(dx)^2 = (1+A^2)(dX_2)^2$; and for OB, $(dx)^2 = (1+A^2)(dX_3)^2$.

3.13. Calculate the *change* in squared length of the line elements of Problem 3.12 and check the result by use of (*3.36*) and the strain tensor $\mathbf{L}_G$ found in Problem 3.11.

Directly from the results of Problem 3.12, the changes are:

(*a*) for OC, $$(dx)^2 - (dX)^2 = (1+A^2)(dX_2^2 + dX_3^2) + 4A\,dX_2\,dX_3 - (dX_2^2 + dX_3^2) = A^2(dX_2^2 + dX_3^2) + 4A\,dX_2\,dX_3$$

(*b*) for OB, $$(dx)^2 - (dX)^2 = (1+A^2)\,dX_3^2 - dX_3^2 = A^2\,dX_3^2$$

(*c*) for OA, $$(dx)^2 - (dX)^2 = (1+A^2)\,dX_2^2 - dX_2^2 = A^2\,dX_2^2.$$

By equation (*3.36*), for OC

$$(dx)^2 - (dX)^2 = [0,\, dX_2,\, dX_3]\begin{bmatrix} 0 & 0 & 0 \\ 0 & A^2 & 2A \\ 0 & 2A & A^2 \end{bmatrix}\begin{bmatrix} 0 \\ dX_2 \\ dX_3 \end{bmatrix} = A^2(dX_2^2 + dX_3^2) + 4A\,dX_2\,dX_3$$

The changes for OA and OB may also be confirmed in the same way.

3.14. For the displacement field of Problem 3.11 calculate the material displacement gradient $\mathbf{J}$ and use this tensor to determine the Lagrangian finite strain tensor $\mathbf{L}_G$. Compare with result of Problem 3.11.

From Problem 3.11 the displacement vector components are $u_1 = 0$, $u_2 = AX_3$, $u_3 = AX_2$ so that

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A \\ 0 & A & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{J}_c \cdot \mathbf{J} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A^2 & 0 \\ 0 & 0 & A^2 \end{pmatrix}$$

Thus from (*3.40*)

$$2\mathbf{L}_G = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A \\ 0 & A & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A \\ 0 & A & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & A^2 & 0 \\ 0 & 0 & A^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A^2 & 2A \\ 0 & 2A & A^2 \end{pmatrix}$$

the identical result obtained in Problem 3.11.

3.15. A displacement field is given by $x_1 = X_1 + AX_2$, $x_2 = X_2 + AX_3$, $x_3 = X_3 + AX_1$ where A is a constant. Calculate the Lagrangian linear strain tensor $\mathbf{L}$ and the Eulerian linear strain tensor $\mathbf{E}$. Compare $\mathbf{L}$ and $\mathbf{E}$ for the case when A is very small.

From (*3.42*),

$$2\mathbf{L} = (\mathbf{J} + \mathbf{J}_c) = \begin{pmatrix} 0 & A & 0 \\ 0 & 0 & A \\ A & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & A \\ A & 0 & 0 \\ 0 & A & 0 \end{pmatrix} = \begin{pmatrix} 0 & A & A \\ A & 0 & A \\ A & A & 0 \end{pmatrix}$$

Inverting the displacement equations gives

$$u_1 = A(A^2x_1 + x_2 - Ax_3)/(1+A^3), \quad u_2 = A(-Ax_1 + A^2x_2 + x_3)/(1+A^3),$$
$$u_3 = A(x_1 - Ax_2 + A^2x_3)/(1+A^3)$$

from which by (*3.43*)

$$2\mathbf{E} = (\mathbf{K} + \mathbf{K}_c) = \frac{A}{1+A^3}\begin{pmatrix} A^2 & 1 & -A \\ -A & A^2 & 1 \\ 1 & -A & A^2 \end{pmatrix} + \frac{A}{1+A^3}\begin{pmatrix} A^2 & -A & 1 \\ 1 & A^2 & -A \\ -A & 1 & A^2 \end{pmatrix}$$

$$= \frac{A}{1+A^3}\begin{pmatrix} 2A^2 & 1-A & 1-A \\ 1-A & 2A^2 & 1-A \\ 1-A & 1-A & 2A^2 \end{pmatrix}$$

When A is very small, A^2 and higher powers may be neglected with the result that $\mathbf{E}$ reduces to $\mathbf{L}$.

3.16. A displacement field is specified by $\mathbf{u} = X_1^2 X_2 \hat{\mathbf{e}}_1 + (X_2 - X_3^2)\hat{\mathbf{e}}_2 + X_2^2 X_3 \hat{\mathbf{e}}_3$. Determine the relative displacement vector $d\mathbf{u}$ in the direction of the $-X_2$ axis at $P(1,2,-1)$. Determine the relative displacements $\mathbf{u}_{Q_i} - \mathbf{u}_P$ for $Q_1(1,1,-1)$, $Q_2(1,3/2,-1)$, $Q_3(1,7/4,-1)$ and $Q_4(1,15/8,-1)$ and compare their directions with the direction of $d\mathbf{u}$.

For the given $\mathbf{u}$, the displacement gradient $\mathbf{J}$ in matrix form is

$$[\partial u_i/\partial X_j] = \begin{bmatrix} 2X_1X_2 & X_1^2 & 0 \\ 0 & 1 & -2X_3 \\ 0 & 2X_2X_3 & X_2^2 \end{bmatrix}$$

so that from *(3.46)* at P in the $-X_2$ direction,

$$[du_i] = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & -4 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$$

Next by direct calculation from $\mathbf{u}$, $\mathbf{u}_P = 2\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 - 4\hat{\mathbf{e}}_3$ and $\mathbf{u}_{Q_1} = \hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_3$. Thus $\mathbf{u}_{Q_1} - \mathbf{u}_P = -\mathbf{e}_1 - \mathbf{e}_2 + 3\,\mathbf{e}_3$. Likewise, $\mathbf{u}_{Q_2} - \mathbf{u}_P = (-\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + 3.5\,\hat{\mathbf{e}}_3)/2$, $\mathbf{u}_{Q_3} - \mathbf{u}_P = (-\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + 3.75\,\hat{\mathbf{e}}_3)/4$, $\mathbf{u}_{Q_4} - \mathbf{u}_P = (-\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + 3.875\,\hat{\mathbf{e}}_3)/8$. It is clear that as Q_i approaches P the direction of the relative displacement of the two particles approaches the limiting direction of $d\mathbf{u}$.

3.17. For the displacement field of Problem 3.16 determine the unit relative displacement vector at $P(1,2,-1)$ in the direction of $Q(4,2,3)$.

The unit vector at P in the direction of Q is $\hat{\boldsymbol{\nu}} = 3\hat{\mathbf{e}}_1/5 + 4\hat{\mathbf{e}}_3/5$, so that from *(3.47)* and the matrix of $\mathbf{J}$ as calculated in Problem 3.16,

$$[du_i/dX] = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & -4 & 4 \end{bmatrix}\begin{bmatrix} 3/5 \\ 0 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 12/5 \\ 8/5 \\ 16/5 \end{bmatrix}$$

3.18. Under the restriction of small deformation theory, $\mathbf{L} = \mathbf{E}$. Accordingly for a displacement field given by $\mathbf{u} = (x_1 - x_3)^2\hat{\mathbf{e}}_1 + (x_2 + x_3)^2\hat{\mathbf{e}}_2 - x_1x_2\hat{\mathbf{e}}_3$, determine the linear strain tensor, the linear rotation tensor and the rotation vector at the point $P(0,2,-1)$.

Here the displacement gradient is given in matrix form by

$$[\partial u_i/\partial x_j] = \begin{bmatrix} 2(x_1 - x_3) & 0 & -2(x_1 - x_3) \\ 0 & 2(x_2 + x_3) & 2(x_2 + x_3) \\ -x_2 & -x_1 & 0 \end{bmatrix}$$

which at the point P becomes

$$[\partial u_i/\partial x_j]_P = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 2 \\ -2 & 0 & 0 \end{bmatrix}$$

Decomposing this matrix into its symmetric and antisymmetric components gives

$$[\epsilon_{ij}] + [\omega_{ij}] = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Therefore from *(3.56)* the rotation vector ω_i has components $\omega_1 = -1$, $\omega_2 = \omega_3 = 0$.

3.19. For the displacement field of Problem 3.18 determine the change in length per unit length (normal strain) in the direction of $\hat{\nu} = (8\hat{e}_1 - \hat{e}_2 + 4\hat{e}_3)/9$ at point $P(0,2,-1)$.

From (3.59) and the strain tensor at P as computed in Problem 3.18, the normal strain at P in the direction of $\hat{\nu}$ is the matrix product

$$\epsilon_p^{(\hat{\nu})} = [8/9, -1/9, 4/9]\begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 8/9 \\ -1/9 \\ 4/9 \end{bmatrix} = -6/81$$

3.20. Show that the change in the right angle between two orthogonal unit vectors $\hat{\nu}$ and $\hat{\mu}$ in the undeformed configuration is given by $\hat{\nu}\cdot 2\mathbf{L}\cdot\hat{\mu}$ for small deformation theory.

Assuming small displacement gradients, the unit vectors in the deformed directions of $\hat{\nu}$ and $\hat{\mu}$ are given by (3.47) as $(\hat{\nu} + \mathbf{J}\cdot\hat{\nu})$ and $(\hat{\mu} + \mathbf{J}\cdot\hat{\mu})$ respectively. (The student should check equations (3.61) and (3.62) by this method.) Writing $\mathbf{J}\cdot\hat{\nu}$ in the equivalent form $\hat{\nu}\cdot\mathbf{J}_c$ and dotting the two displaced unit vectors gives (as in (3.63)), $\cos\theta = \sin(\pi/2 - \theta) = \sin\gamma_{\nu\mu} = \gamma_{\nu\mu}$ or $\gamma_{\nu\mu} = [\hat{\nu} + \hat{\nu}\cdot\mathbf{J}_c]\cdot[\hat{\mu} + \mathbf{J}\cdot\hat{\mu}] = \hat{\nu}\cdot\hat{\mu} + \hat{\nu}\cdot(\mathbf{J}+\mathbf{J}_c)\cdot\hat{\mu} + \hat{\nu}\cdot\mathbf{J}_c\cdot\mathbf{J}\cdot\hat{\mu}$. Here $\mathbf{J}_c\cdot\mathbf{J}$ is of higher order for small displacement gradients and since $\hat{\nu}\perp\hat{\mu}$, $\hat{\nu}\cdot\hat{\mu} = 0$ so that finally by (3.42), $\gamma_{\nu\mu} = \hat{\nu}\cdot 2\mathbf{L}\cdot\hat{\mu}$.

3.21. Use the results of Problem 3.20 to compute the change in the right angle between $\hat{\nu} = (8\hat{e}_1 - \hat{e}_2 + 4\hat{e}_3)/9$ and $\hat{\mu} = (4\hat{e}_1 + 4\hat{e}_2 - 7\hat{e}_3)/9$ at the point $P(0,2,-1)$ for the displacement field of Problem 3.18.

Since $\mathbf{L} = \mathbf{E}$ for small deformation theory, the strain tensor $\epsilon_{ij} = l_{ij}$ and so at P

$$\gamma_{\nu\mu} = [8/9, -1/9, 4/9]\begin{bmatrix} 4 & 0 & -4 \\ 0 & 4 & 2 \\ -4 & 2 & 0 \end{bmatrix}\begin{bmatrix} 4/9 \\ 4/9 \\ -7/9 \end{bmatrix} = 318/81$$

STRETCH AND ROTATION (Sec. 3.10-3.11)

3.22. For the shear deformation $x_1 = X_1$, $x_2 = X_2 + AX_3$, $x_3 = X_3 + AX_2$ of Problem 3.11 show that the stretch $\Lambda_{(\hat{m})}$ is unity (zero normal strain) for line elements parallel to the X_1 axis. For the diagonal directions OC and DB of the infinitesimal square $OBCD$ (Fig. 3-14), compute $\Lambda_{(\hat{m})}$ and check the results by direct calculation from the displacement field.

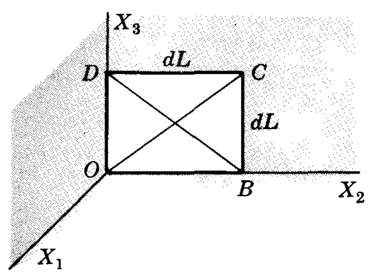


Fig. 3-14

From (3.66) and the matrix of $\mathbf{G}$ as determined in Problem 3.11, the squared stretch for $\hat{m} = \hat{e}_1$ is

$$\Lambda^2_{(\hat{e}_1)} = [1, 0, 0]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+A^2 & 2A \\ 0 & 2A & 1+A^2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1$$

Likewise for OC, $\hat{m} = (\hat{e}_2 + \hat{e}_3)/\sqrt{2}$ and so

$$\Lambda^2_{(\hat{m})} = [0, 1/\sqrt{2}, 1/\sqrt{2}]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+A^2 & 2A \\ 0 & 2A & 1+A^2 \end{bmatrix}\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = (1+A)^2$$

From the displacement equations the deformed location of C is $x_1 = 0$, $x_2 = dL + A\,dL$, $x_3 = dL + A\,dL$. Thus $(dx)^2 = 2(1+A)^2(dL)^2$ and since $dX = \sqrt{2}\,dL$, the squared stretch $(dx/dX)^2$ is $(1+A)^2$ as calculated from (3.66).

Similarly, for DB, $\hat{\mathbf{m}} = (-\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3)/\sqrt{2}$ and so $\Lambda^2_{(\hat{\mathbf{m}})} = (1-A)^2$.

3.23. The stretch ratios $\Lambda_{(\hat{\mathbf{m}})}$ and $\lambda_{(\hat{\mathbf{n}})}$ are equal only if $\hat{\mathbf{n}}$ is the deformed direction of $\hat{\mathbf{m}}$. For the displacement field of Problem 3.22, calculate $\lambda_{(\hat{\mathbf{n}})}$ for $\hat{\mathbf{n}} = (\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3)/\sqrt{2}$ and show that it agrees with $\Lambda^2_{(\hat{\mathbf{m}})}$ for the diagonal OC in Problem 3.22.

Inverting the displacement equations of Problem 3.22 one obtains

$$X_1 = x_1, \quad X_2 = (x_2 - Ax_3)/(1-A^2), \quad X_3 = (x_3 - Ax_2)/(1-A^2)$$

from which the Cauchy deformation tensor $\mathbf{C}$ may be computed. Then using (3.67),

$$\frac{1}{\lambda^2_{(\hat{\mathbf{n}})}} = [0, 1/\sqrt{2}, 1/\sqrt{2}]\begin{bmatrix} 1 & 0 & 0 \\ 0 & (1+A^2)/(1-A^2)^2 & -2A/(1-A^2)^2 \\ 0 & -2A/(1-A^2)^2 & (1+A^2)/(1-A^2)^2 \end{bmatrix}\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = (1-A)^2/(1-A^2)^2$$

Thus $\lambda^2_{(\hat{\mathbf{n}})} = (1-A^2)^2/(1-A)^2 = (1+A)^2$ which is identical with $\Lambda^2_{(\hat{\mathbf{m}})}$ calculated for OC. The diagonal element OC does not change direction under the given shear deformation.

3.24. By a polar decomposition of the deformation gradient $\mathbf{F}$ for the shear deformation $x_1 = X_1$, $x_2 = X_2 + AX_3$, $x_3 = X_3 + AX_2$, determine the right stretch tensor $\mathbf{S}$ together with the rotation tensor $\mathbf{R}$. Show that the principal values of $\mathbf{S}$ are the stretch ratios of the diagonals OC and DB determined in Problem 3.22.

In the polar decomposition of $\mathbf{F}$, the stretch tensor $\mathbf{S} = \sqrt{\mathbf{G}}$; and from (3.73), $\mathbf{R} = \mathbf{F}\mathbf{S}^{-1}$. By (3.35), $\mathbf{G} = \mathbf{F}_c \cdot \mathbf{F}$ or here $[G_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+A^2 & 2A \\ 0 & 2A & 1+A^2 \end{bmatrix}$. The principal axes of $\mathbf{G}$ are given by a 45° rotation about X_1 with the tensor in principal form $[G^*_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1-A)^2 & 0 \\ 0 & 0 & (1+A)^2 \end{bmatrix}$.

$$\text{Therefore} \quad [S_{ij}] = [\sqrt{G^*_{ij}}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1-A) & 0 \\ 0 & 0 & (1+A) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \Lambda_{(DB)} & 0 \\ 0 & 0 & \Lambda_{(OC)} \end{bmatrix}$$

Relative to the coordinate axes X_i, the decomposition is

$$[F_{ij}] = [R_{ik}][S_{kj}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & A \\ 0 & A & 1 \end{bmatrix}$$

In this example the deformation gradient $\mathbf{F}$ is its own stretch tensor $\mathbf{S}$ and $\mathbf{R} = \mathbf{I}$. This is the result of the coincidence of the principal axes of $\mathbf{L}_G$ and $\mathbf{E}_A$ for the given shear deformation.

3.25. An infinitesimal rigid body rotation is given by $u_1 = -CX_2 + BX_3$, $u_2 = CX_1 - AX_3$, $u_3 = -BX_1 + AX_2$ where A, B, C are very small constants. Show that the stretch is zero ($\mathbf{S} = \mathbf{I}$) if terms involving squares and products of the constants are neglected.

For this displacement,

$$[G_{ij}] = \begin{bmatrix} 1+C^2+B^2 & -AB & -AC \\ -AB & 1+A^2+C^2 & -BC \\ -AC & -BC & 1+A^2+B^2 \end{bmatrix}$$

Neglecting higher order terms, this becomes

$$[G_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [\sqrt{G_{ij}}\,] = [S_{ij}]$$

STRAIN TRANSFORMATIONS AND PRINCIPAL STRAINS (Sec. 3.12-3.14)

3.26. For the shear deformation $x_1 = X_1$, $x_2 = X_2 + \sqrt{2}\,X_3$, $x_3 = X_3 + \sqrt{2}\,X_2$ show that the principal directions of $\mathbf{L}_G$ and $\mathbf{E}_A$ coincide as was asserted in Problem 3.24.

From (*3.37*), $[L_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \sqrt{2} \\ 0 & \sqrt{2} & 1 \end{bmatrix}$ which for principal axes given by the transformation matrix $[a_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ becomes $[L^*_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1-\sqrt{2} & 0 \\ 0 & 0 & 1+\sqrt{2} \end{bmatrix}$.

Likewise from (*3.39*), $[E_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & \sqrt{2} \\ 0 & \sqrt{2} & -1 \end{bmatrix}$ which by the same transformation matrix $[a_{ij}]$ is converted into the principal-axes form $[E^*_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1-\sqrt{2} & 0 \\ 0 & 0 & -1+\sqrt{2} \end{bmatrix}$. The student should verify these calculations.

3.27. Using the definition (*3.37*), show that the Lagrangian finite strain tensor L_{ij} transforms as a second order Cartesian tensor under the coordinate transformations $x_i = b_{ji}x'_j$ and $X'_i = b_{ij}X_j$.

By (*3.37*), $L_{ij} = \dfrac{1}{2}\left(\dfrac{\partial x_k}{\partial X_i}\dfrac{\partial x_k}{\partial X_j} - \delta_{ij}\right)$ which by the stated transformation becomes

$$\begin{aligned} L_{ij} &= \frac{1}{2}\left(\frac{\partial(b_{pk}x'_p)}{\partial X'_m}\frac{\partial X'_m}{\partial X_i}\frac{\partial(b_{qk}x'_q)}{\partial X'_n}\frac{\partial X'_n}{\partial X_j} - \frac{\partial x_i}{\partial x_j}\right) \\ &= \frac{1}{2}\left(b_{mi}b_{nj}\delta_{pq}\frac{\partial x'_p}{\partial X'_m}\frac{\partial x'_q}{\partial X'_n} - \frac{\partial(b_{mi}x'_m)}{\partial x'_n}\frac{\partial x'_n}{\partial x_j}\right) \qquad (\text{since } b_{pk}b_{qk} = \delta_{pq}) \\ &= b_{mi}b_{nj}\left[\frac{1}{2}\left(\frac{\partial x'_p}{\partial X'_m}\frac{\partial x'_p}{\partial X'_n} - \delta'_{mn}\right)\right] = b_{mi}b_{nj}L'_{mn} \end{aligned}$$

3.28. A certain homogeneous deformation field results in the finite strain tensor $[L_{ij}] = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 1 & -2 \\ -2 & -2 & 6 \end{bmatrix}$. Determine the principal strains and their directions for this deformation.

Being a symmetric second order Cartesian tensor, the principal strains are the roots of

$$\begin{vmatrix} 1-L & 3 & -2 \\ 3 & 1-L & -2 \\ -2 & -2 & 6-L \end{vmatrix} = L^3 - 8L^2 - 4L + 32 = 0$$

Thus $L_{(1)} = -2$, $L_{(2)} = 2$, $L_{(3)} = 8$. The transformation matrix for principal directions is

$$[a_{ij}] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}$$

3.29. For the homogeneous deformation $x_1 = \sqrt{3}X_1$, $x_2 = 2X_2$, $x_3 = \sqrt{3}X_3 - X_2$ determine the material strain ellipsoid resulting from deformation of the spherical surface $X_1^2 + X_2^2 + X_3^2 = 1$. Show that this ellipsoid has the form $x_1^2/\Lambda_{(1)}^2 + x_2^2/\Lambda_{(2)}^2 + x_3^2/\Lambda_{(3)}^2 = 1$.

By *(3.82)*, or alternatively by inverting the given displacement equations and substituting into $X_iX_i = 1$, the material strain ellipsoid is $x_1^2 + x_2^2 + x_3^2 + x_2x_3 = 3$. This equation is put into the principal-axes form $x_1^2/3 + x_2^2/6 + x_3^2/2 = 1$ by the transformation

$$[a_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

From the deformation equations, the stretch tensor $\mathbf{S} = \sqrt{\mathbf{G}}$ is given (calculation is similar to that in Problem 3.24) as

$$[S_{ij}] = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \dfrac{3\sqrt{3}+1}{2\sqrt{2}} & \dfrac{\sqrt{3}-3}{2\sqrt{2}} \\ 0 & \dfrac{\sqrt{3}-3}{2\sqrt{2}} & \dfrac{+\sqrt{3}+3}{2\sqrt{2}} \end{bmatrix}$$

which by the transformation $[a_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix}$ is put into the principal form

$$[S_{ij}^*] = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

with principal stretches $\Lambda_{(1)}^2 = 3$, $\Lambda_{(2)}^2 = 6$, $\Lambda_{(3)}^2 = 2$. Note also that the principal stretches may be calculated directly from *(3.66)* using $[a_{ij}]$ above.

3.30. For the deformation of Problem 3.29, determine the spatial strain ellipsoid and show that it is of the form $\Lambda_{(1)}^2 X_1^2 + \Lambda_{(2)}^2 X_2^2 + \Lambda_{(3)}^2 X_3^2 = 1$.

By *(3.84)* the sphere $x_ix_i = 1$ resulted from the ellipsoid $\mathbf{X} \cdot \mathbf{G} \cdot \mathbf{X} = 1$, or

$$[X_1, X_2, X_3] \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & -\sqrt{3} \\ 0 & -\sqrt{3} & 3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 3X_1^2 + 5X_2^2 + 3X_3^2 - 2\sqrt{3}X_2X_3 = 1$$

This ellipsoid is put into the principal-axes form $3X_1^2 + 6X_2^2 + 2X_3^2 = 1$ by the transformation

$$[a_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \\ 0 & -1/2 & \sqrt{3}/2 \end{bmatrix}$$

3.31. Verify by direct expansion that the second invariant II_L of the strain tensor may be expressed by

$$II_L = \begin{vmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{vmatrix} + \begin{vmatrix} l_{11} & l_{13} \\ l_{31} & l_{33} \end{vmatrix} + \begin{vmatrix} l_{22} & l_{23} \\ l_{32} & l_{33} \end{vmatrix}$$

Expansion of the given determinants results in $II_L = l_{11}l_{22} + l_{22}l_{33} + l_{33}l_{11} - (l_{12}^2 + l_{23}^2 + l_{31}^2)$. In comparison, direct expansion of the second equation of (*3.91*) yields

$$\begin{aligned} II_L &= \tfrac{1}{2}[(l_{11}+l_{22}+l_{33})l_{jj} - (l_{1j}l_{1j} + l_{2j}l_{2j} + l_{3j}l_{3j})] \\ &= \tfrac{1}{2}[(l_{11}+l_{22}+l_{33})(l_{11}+l_{22}+l_{33}) - (l_{11}l_{11} + l_{12}l_{12} + l_{13}l_{13} \\ &\qquad + l_{21}l_{21} + l_{22}l_{22} + l_{23}l_{23} + l_{31}l_{31} + l_{32}l_{32} + l_{33}l_{33})] \\ &= l_{11}l_{22} + l_{22}l_{33} + l_{33}l_{11} - (l_{12}^2 + l_{23}^2 + l_{31}^2) \end{aligned}$$

3.32. For the finite homogeneous deformation given by $u_i = A_{ij}X_j$ where A_{ij} are constants, determine an expression for the change of volume per unit original volume. If the A_{ij} are very small, show that the result reduces to the cubical dilatation.

Consider a rectangular element of volume having original dimensions dX_1, dX_2, dX_3 along the coordinate axes. For the given deformation, $x_i = (A_{ij} + \delta_{ij})X_j$. Thus by (*3.33*) the original volume dV_0 becomes a skewed parallelepiped having edge lengths $dx_i = (A_{i(n)} + \delta_{i(n)})\,dX_{(n)}$, $n = 1,2,3$. From (*1.109*) this deformed element has the volume $dV = \epsilon_{ijk}(A_{i1}+\delta_{i1})(A_{j2}+\delta_{j2})(A_{k3}+\delta_{k3})\,dX_1\,dX_2\,dX_3$. Then

$$\frac{dV}{dV_0} = \frac{dV_0 + \Delta V}{dV_0} = 1 + \frac{\Delta V}{dV_0} = \epsilon_{ijk}(A_{i1}+\delta_{i1})(A_{j2}+\delta_{j2})(A_{k3}+\delta_{k3})$$

If the A_{ij} are very small and their powers neglected,

$$\Delta V/dV_0 = \epsilon_{ijk}(A_{i1}\delta_{j2}\delta_{k3} + \delta_{i1}A_{j2}\delta_{k3} + \delta_{i1}\delta_{j2}A_{k3} + \delta_{i1}\delta_{j2}\delta_{k3}) - 1 = A_{11} + A_{22} + A_{33}$$

For linear theory the cubical dilatation $l_{ii} = \partial u_i/\partial X_i$, which for $u_i = A_{ij}X_j$ is $l_{ii} = A_{11} + A_{22} + A_{33}$.

3.33. A linear (small strain) deformation is specified by $u_1 = 4x_1 - x_2 + 3x_3$, $u_2 = x_1 + 7x_2$, $u_3 = -3x_1 + 4x_2 + 4x_3$. Determine the principal strains $\epsilon_{(n)}$ and the principal deviator strains $e_{(n)}$ for this deformation.

Since ϵ_{ij} is the symmetrical part of the displacement gradient $\partial u_i/\partial x_j$, it is given here by

$$\epsilon_{ij} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 7 & 2 \\ 0 & 2 & 4 \end{pmatrix}$$ or in principal-axes form by $$\epsilon^*_{ij} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$ Also, $\epsilon_{kk}/3 = 5$ and so the strain deviator is $$e_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & -1 \end{pmatrix}$$ and its principal-axes form $$e^*_{ij} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$ Note that $e_{(m)} = \epsilon_{(m)} - \epsilon_{kk}/3$.

PLANE STRAIN AND COMPATIBILITY (Sec. 3.15-3.16)

3.34. A 45° strain-rosette measures longitudinal strain along the axes shown in Fig. 3-15. At a point P, $\epsilon_{11} = 5 \times 10^{-4}$, $\epsilon'_{11} = 4 \times 10^{-4}$, $\epsilon_{22} = 7 \times 10^{-4}$ in/in. Determine the shear strain ϵ_{12} at the point.

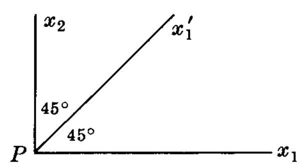


Fig. 3-15

By (*3.59*), with $\hat{\nu} = (\hat{e}_1 + \hat{e}_2)/\sqrt{2}$ as the unit vector in the x'_1 direction,

$$[1/\sqrt{2}, 1/\sqrt{2}, 0]\begin{bmatrix} 5\times 10^{-4} & \epsilon_{12} & 0 \\ \epsilon_{12} & 7\times 10^{-4} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = 4\times 10^{-4}$$

Therefore $\dfrac{12\times 10^{-4} + 2\epsilon_{12}}{2} = 4\times 10^{-4}$ or $\epsilon_{12} = -2\times 10^{-4}$.

3.35. Construct the Mohr's circles for the case of plane strain

$$\epsilon_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 5 & \sqrt{3} \\ 0 & \sqrt{3} & 3 \end{pmatrix}$$

and determine the maximum shear strain. Verify the result analytically.

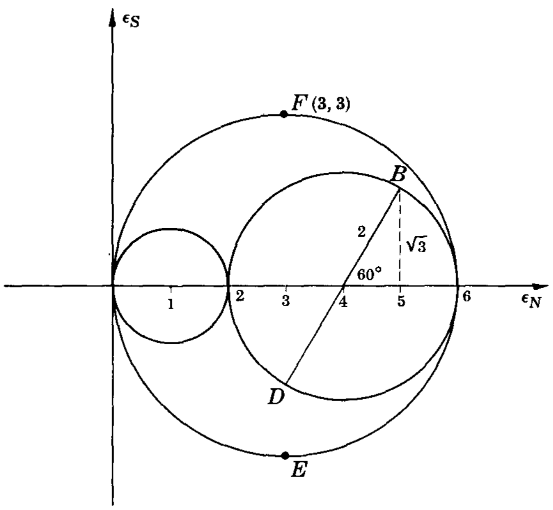


Fig. 3-16

With the given state of strain referred to the x_i axes, the points $B(\epsilon_{22} = 5, \epsilon_{23} = \sqrt{3})$ and D are established as the diameter of the larger inner circle in Fig. 3-16. Since $\epsilon_{(1)} = 0$ is a principal value for plane strain, the other circles are drawn as shown.

A rotation of 30° about the x_1 axis (equivalent to 60° in the Mohr's diagram) results in the principal strain axes with the principal strain tensor ϵ^*_{ij} given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \\ 0 & -1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & \sqrt{3} \\ 0 & \sqrt{3} & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

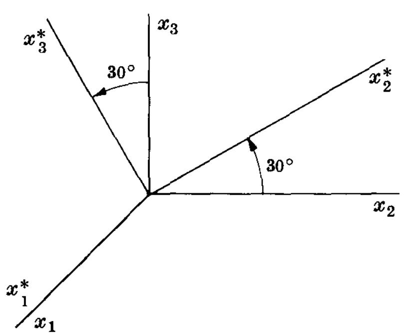


Fig. 3-17

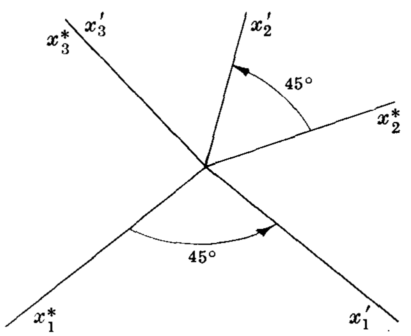


Fig. 3-18

Next a rotation of 45° about the x_3^* axis (90° in the Mohr diagram) results in the x_i' axes and the associated strain tensor ϵ'_{ij} given by

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

the first two rows of which represent the state of strain specified by point F in Fig. 3-16. Note that a rotation of −45° about x_3^* would correspond to point E in Fig. 3-16.

3.36. The state of strain throughout a continuum is specified by

$$\epsilon_{ij} = \begin{pmatrix} x_1^2 & x_2^2 & x_1x_3 \\ x_2^2 & x_3 & x_3^2 \\ x_1x_3 & x_3^2 & 5 \end{pmatrix}$$

Are the compatibility equations for strain satisfied?

Substituting directly into (*3.104*), all equations are satisfied identically. The student should carry out the details.

MISCELLANEOUS PROBLEMS

3.37. Derive the indicial form of the Lagrangian finite strain tensor L_G of (*3.40*) from its definition (*3.37*).

From (*3.24*), $\partial x_i/\partial X_j = \delta_{ij} + \partial u_i/\partial X_j$. Thus (*3.37*) may be written

$$L_{ij} = \frac{1}{2}\left[\left(\delta_{ki} + \frac{\partial u_k}{\partial X_i}\right)\left(\delta_{kj} + \frac{\partial u_k}{\partial X_j}\right) - \delta_{ij}\right]$$

$$= \frac{1}{2}\left[\delta_{ki}\delta_{kj} + \delta_{ki}\frac{\partial u_k}{\partial X_j} + \delta_{kj}\frac{\partial u_k}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j} - \delta_{ij}\right]$$

$$= \frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right]$$

3.38. A displacement field is defined by $x_1 = X_1 - CX_2 + BX_3$, $x_2 = CX_1 + X_2 - AX_3$, $x_3 = -BX_1 + AX_2 + X_3$. Show that this displacement represents a rigid body rotation only if the constants A, B, C are very small. Determine the rotation vector $\mathbf{w}$ for the infinitesimal rigid body rotation.

For the given displacements, $\mathbf{F} = \begin{pmatrix} 1 & -C & B \\ C & 1 & -A \\ -B & A & 1 \end{pmatrix}$ and from (*3.37*),

$$\mathbf{L}_G = \frac{1}{2}\begin{pmatrix} B^2+C^2 & -AB & -AC \\ -AB & A^2+C^2 & -BC \\ -AC & -BC & A^2+B^2 \end{pmatrix}$$

If products of the constants are neglected, this strain tensor is zero and the displacement reduces to a rigid body rotation. From (*3.50*), the rotation vector is

$$\mathbf{w} = \frac{1}{2}\begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \partial/\partial X_1 & \partial/\partial X_2 & \partial/\partial X_3 \\ -CX_2+BX_3 & CX_1-AX_3 & -BX_1+AX_2 \end{vmatrix} = A\hat{\mathbf{e}}_1 + B\hat{\mathbf{e}}_2 + C\hat{\mathbf{e}}_3$$

3.39. For the rigid body rotation represented by $u_1 = 0.02X_3$, $u_2 = -0.03X_3$, $u_3 = -0.02X_1 + 0.03X_2$, determine the relative displacement of $Q(3, 0.1, 4)$ with respect to $P(3, 0, 4)$.

From the displacement equations, $\mathbf{u}_Q = .08\,\hat{\mathbf{e}}_1 - .12\,\hat{\mathbf{e}}_2 - .057\,\hat{\mathbf{e}}_3$ and $\mathbf{u}_P = .08\,\hat{\mathbf{e}}_1 - .12\,\hat{\mathbf{e}}_2 - .06\,\hat{\mathbf{e}}_3$. Hence $d\mathbf{u} = \mathbf{U}_Q - \mathbf{U}_P = -.003\,\hat{\mathbf{e}}_3$. The same result is obtained by (*3.51*), with $\mathbf{w} = .03\,\hat{\mathbf{e}}_1 + .02\,\hat{\mathbf{e}}_2$:

$$d\mathbf{u} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ .03 & .02 & 0 \\ 0 & .1 & 0 \end{vmatrix} = -.003\,\hat{\mathbf{e}}_3$$

3.40. For a state of plane strain parallel to the x_2x_3 axes, determine expressions for the normal strain ϵ'_{22} and the shear strain ϵ'_{23} when the primed and unprimed axes are oriented as shown in Fig. 3-19.

By equation *(3.59)*,

$$\epsilon'_{22} = [0, \cos\theta, \sin\theta]\begin{bmatrix} 0 & 0 & 0 \\ 0 & \epsilon_{22} & \epsilon_{23} \\ 0 & \epsilon_{23} & \epsilon_{33} \end{bmatrix}\begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix}$$

$$= \epsilon_{22}\cos^2\theta + 2\epsilon_{23}\sin\theta\cos\theta + \epsilon_{33}\sin^2\theta$$

$$= \frac{\epsilon_{22}+\epsilon_{33}}{2} + \frac{\epsilon_{22}-\epsilon_{33}}{2}\cos 2\theta + \epsilon_{23}\sin 2\theta$$

Similarly from *(3.65)* and Problem 3.20,

$$\epsilon'_{23} = [0, \cos\theta, \sin\theta]\begin{bmatrix} 0 & 0 & 0 \\ 0 & \epsilon_{22} & \epsilon_{23} \\ 0 & \epsilon_{23} & \epsilon_{33} \end{bmatrix}\begin{bmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$= -\epsilon_{22}\sin\theta\cos\theta + \epsilon_{23}\cos^2\theta - \epsilon_{23}\sin^2\theta + \epsilon_{33}\sin\theta\cos\theta$$

$$= \epsilon_{23}\cos 2\theta - \frac{\epsilon_{22}-\epsilon_{33}}{2}\sin 2\theta$$

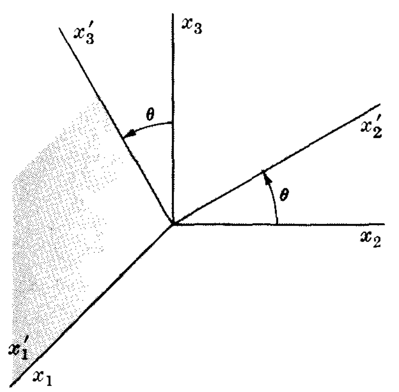


Fig. 3-19

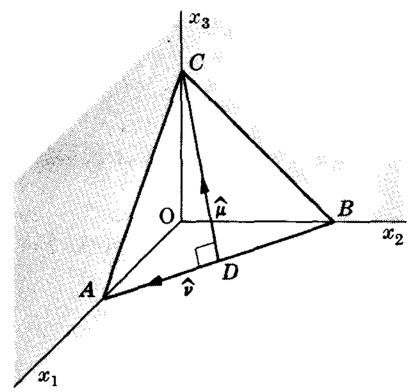


Fig. 3-20

3.41. For a homogeneous deformation the small strain tensor is given by

$$[\epsilon_{ij}] = \begin{bmatrix} 0.01 & -0.005 & 0 \\ -0.005 & 0.02 & 0.01 \\ 0 & 0.01 & -0.03 \end{bmatrix}$$

What is the change in the 90° angle ADC depicted by the small tetrahedron $OABC$ in Fig. 3-20 if $OA = OB = OC$, and D is the midpoint of AB?

The unit vectors $\hat{\nu}$ and $\hat{\mu}$ at D are given by $\hat{\nu} = (\hat{e}_1 - \hat{e}_2)/\sqrt{2}$ and $\hat{\mu} = (2\hat{e}_3 - \hat{e}_2 - \hat{e}_1)/\sqrt{6}$. From the result of Problem 3.20,

$$\gamma_{\mu\nu} = [1/\sqrt{2}, -1/\sqrt{2}, 0]\begin{bmatrix} .02 & -.01 & 0 \\ -.01 & .04 & .02 \\ 0 & .02 & -.06 \end{bmatrix}\begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} = -.01/\sqrt{3}$$

3.42. At a point the strain tensor is given by $\epsilon_{ij} = \begin{pmatrix} 5 & -1 & -1 \\ -1 & 4 & 0 \\ -1 & 0 & 4 \end{pmatrix}$ and in principal form by $\epsilon^*_{ij} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Calculate the strain invariants for each of these tensors and show their equivalence.

By (*3.95*) and Problem 3.31, $I_E = 5+4+4 = 13$, $I_{E^*} = 6+4+3 = 13$. Likewise $II_E = 19+19+16 = 54$, $II_{E^*} = 24+18+12 = 54$. Finally $III_E = 5(16) - 4 - 4 = 72$, $III_{E^*} = (6)(4)(3) = 72$. The student should check these calculations.

3.43. For the displacement field $x_1 = X_1 + AX_3$, $x_2 = X_2 - AX_3$, $x_3 = X_3 - AX_1 + AX_2$, determine the finite strain tensor L_G. Show that if A is very small the displacement represents a rigid body rotation.

Since $u_1 = AX_3$, $u_2 = -AX_3$, $u_3 = -AX_1 + AX_2$, by (*3.40*),

$$2L_G = \begin{pmatrix} 0 & 0 & A \\ 0 & 0 & -A \\ -A & A & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -A \\ 0 & 0 & A \\ A & -A & 0 \end{pmatrix} + \begin{pmatrix} A^2 & -A^2 & 0 \\ -A^2 & A^2 & 0 \\ 0 & 0 & 2A^2 \end{pmatrix} = \begin{pmatrix} A^2 & -A^2 & 0 \\ -A^2 & A^2 & 0 \\ 0 & 0 & 2A^2 \end{pmatrix}$$

If A is small so that A^2 may be neglected, $L_G \equiv 0$; and by (*3.50*) the rotation vector $\mathbf{w} = A\,\hat{\mathbf{e}}_1 + A\,\hat{\mathbf{e}}_2$.

3.44. Show that the displacement field $u_1 = Ax_1 + 3x_2$, $u_2 = 3x_1 - Bx_2$, $u_3 = 5$ gives a state of plane strain and determine the relationship between A and B for which the deformation is *isochoric* (constant volume deformation).

From the displacement equations, by (*3.43*), $\epsilon_{ij} = \begin{bmatrix} A & 3 & 0 \\ 3 & -B & 0 \\ 0 & 0 & 0 \end{bmatrix}$ which is of the form of (*3.99*). From (*3.96*), the cubical dilatation is $D \equiv \epsilon_{ii} = A - B$, which is zero if $A = B$.

3.45. A so-called delta-rosette for measuring longitudinal surface strains has the shape of the equilateral triangle Δ and records normal strains $\epsilon_{11}, \epsilon'_{11}, \epsilon''_{11}$ in the directions shown in Fig. 3-21. If $\epsilon_{11} = a$, $\epsilon'_{11} = b$, $e''_{11} = c$, determine ϵ_{12} and ϵ_{22} at the point.

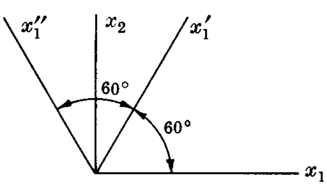


Fig. 3-21

By (*3.59*) with $L = E$, for the x_1' direction,

$$[1/2, \sqrt{3}/2, 0] \begin{bmatrix} a & \epsilon_{12} & 0 \\ \epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \\ 0 \end{bmatrix} = b \quad \text{or} \quad 2\sqrt{3}\,\epsilon_{12} + 3\epsilon_{22} = 4b - a$$

Also for the x_1'' direction

$$[-1/2, \sqrt{3}/2, 0] \begin{bmatrix} a & \epsilon_{12} & 0 \\ \epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \\ 0 \end{bmatrix} = c \quad \text{or} \quad -2\sqrt{3}\,\epsilon_{12} + 3\epsilon_{22} = 4c - a$$

Solving simultaneously for ϵ_{12} and ϵ_{22} yields $\epsilon_{12} = (b-c)/\sqrt{3}$ and $\epsilon_{22} = (-a + 2b + 2c)/3$.

3.46. Derive equation (*3.72*) expressing the change in angle between the coordinate directions X_2 and X_3 under a finite deformation. Show that (*3.72*) reduces to (*3.65*) when displacement gradients are small.

Let $\gamma_{23} = \pi/2 - \theta$ be the angle change as shown in Fig. 3-5. Then $\sin\gamma_{23} = \cos(\pi/2 - \theta) = \hat{\mathbf{n}}_2 \cdot \hat{\mathbf{n}}_3$, or by (*3.33*) and (*3.34*)

$$\sin\gamma_{23} = \frac{dx_2}{|dx_2|} \cdot \frac{dx_3}{|dx_2|} = \frac{d\mathbf{X}_2 \cdot \mathsf{F}_c \cdot \mathsf{F} \cdot d\mathbf{X}_3}{\sqrt{d\mathbf{X}_2 \cdot \mathsf{G} \cdot d\mathbf{X}_2}\,\sqrt{d\mathbf{X}_3 \cdot \mathsf{G} \cdot d\mathbf{X}_3}}$$

Now dividing the numerator and denominator of this equation by $|d\mathbf{X}_2|$ and $|d\mathbf{X}_3|$ and using (*3.35*) and (*3.66*) gives

$$\sin\gamma_{23} = \frac{\hat{\mathbf{e}}_2 \cdot \mathsf{G} \cdot \hat{\mathbf{e}}_3}{\sqrt{\hat{\mathbf{e}}_2 \cdot \mathsf{G} \cdot \hat{\mathbf{e}}_2}\,\sqrt{\hat{\mathbf{e}}_3 \cdot \mathsf{G} \cdot \hat{\mathbf{e}}_3}} = \frac{\hat{\mathbf{e}}_2 \cdot \mathsf{G} \cdot \hat{\mathbf{e}}_3}{\Lambda_{(\hat{\mathbf{e}}_2)}\Lambda_{(\hat{\mathbf{e}}_3)}}$$

Next from (*3.37*), $\hat{\mathbf{e}}_2 \cdot \mathsf{G} \cdot \hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_2 \cdot (2\mathsf{L}_G + \mathsf{I}) \cdot \hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_2 \cdot 2\mathsf{L}_G \cdot \hat{\mathbf{e}}_3 + \hat{\mathbf{e}}_2 \cdot \mathsf{I} \cdot \hat{\mathbf{e}}_3 = 2L_{23}$ since $\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_3 = 0$. Also from (*3.68*), $\Lambda_{(\hat{\mathbf{e}}_2)} = \sqrt{1 + 2L_{22}}$, etc., and so

$$\sin\gamma_{23} = \frac{2L_{23}}{\sqrt{1+2L_{22}}\,\sqrt{1+2L_{33}}}$$

$$= \frac{\partial u_2/\partial X_3 + \partial u_3/\partial X_2 + (\partial u_k/\partial X_2)(\partial u_k/\partial X_3)}{\sqrt{1 + 2\partial u_2/\partial X_2 + (\partial u_k/\partial X_2)(\partial u_k/\partial X_2)}\,\sqrt{1 + 2\partial u_3/\partial X_3 + (\partial u_k/\partial X_3)(\partial u_k/\partial X_3)}}$$

If $\partial u_i/\partial X_j \ll 1$ this reduces to $\sin\gamma_{23} = \partial u_2/\partial X_3 + \partial u_3/\partial X_2 = 2l_{23}$.

3.47. For the simple shear displacement $x_1 = X_1$, $x_2 = X_2$, $x_3 = X_3 + 2X_2/\sqrt{3}$, determine the direction of the line element in the X_2X_3 plane for which the normal strain is zero.

Let $\hat{\mathbf{m}} = m_2\hat{\mathbf{e}}_2 + m_3\hat{\mathbf{e}}_3$ be the unit normal in the direction of zero strain. Then from (*3.66*), since $\Lambda^2_{(\hat{\mathbf{m}})} = 1$,

$$[0, m_2, m_3]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7/3 & 2/\sqrt{3} \\ 0 & 2/\sqrt{3} & 1 \end{bmatrix}\begin{bmatrix} 0 \\ m_2 \\ m_3 \end{bmatrix} = 1$$

or $7m_2^2 + 4\sqrt{3}\,m_2m_3 + 3m_3^2 = 3$. Also $m_2^2 + m_3^2 = 1$, and solving simultaneously $m_2 = \pm\sqrt{3}/2$, $m_3 = \mp 1/2$, or $m_2 = 0$, $m_3 = \pm 1$. Thus there is zero strain along the X_3 axis and for the element at 60° to the X_3 axis.

The student should verify this result by using the relation $\hat{\mathbf{m}} \cdot 2\mathsf{L}_G \cdot \hat{\mathbf{m}} = 0$ derived from (*3.36*).

Supplementary Problems

3.48. For the shear displacement of Problem 3.47, determine the equation of the ellipse into which the circle $X_2^2 + X_3^2 = 1$ is deformed. *Ans.* $x_2^2 + 9x_3^2 = 3$

3.49. Determine the shear angle γ_{23} for the deformation of Problem 3.47 (Fig. 3-22). *Ans.* $\gamma_{23} = \sin^{-1} 2/\sqrt{7}$

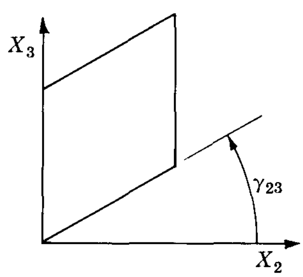


Fig. 3-22

3.50. Given the displacement field $x_1 = X_1 + 2X_3$, $x_2 = X_2 - 2X_3$, $x_3 = X_3 - 2X_1 + 2X_2$, determine the Lagrangian and Eulerian finite strain tensors L_G and E_A.

Ans. $\mathsf{L}_G = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$, $\mathsf{E}_A = \frac{1}{9}\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

3.51. Determine the principal-axes form of the two tensors of Problem 3.50.

$$Ans.\quad L_G^* = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad E_A^* = \begin{pmatrix} 4/9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4/9 \end{pmatrix}$$

3.52. For the displacement field of Problem 3.50 determine the deformation gradient $\mathbf{F}$, and by a polar decomposition of $\mathbf{F}$ find the rotation tensor $\mathbf{R}$ and the right stretch tensor $\mathbf{S}$.

$$Ans.\quad \mathbf{R} = \frac{1}{3}\begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ -2 & 2 & 1 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ -2 & 2 & 1 \end{pmatrix}$$

3.53. Show that the first invariant of L_G may be written in terms of the principal stretches as $I_{L_G} = [(\Lambda^2_{(\hat{e}_1)} - 1) + (\Lambda^2_{(\hat{e}_2)} - 1) + (\Lambda^2_{(\hat{e}_3)} - 1)]/2$. *Hint:* See equation (*3.68*).

3.54. The strain tensor at a point is given by $\epsilon_{ij} = \begin{pmatrix} 1 & -3 & \sqrt{2} \\ -3 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 4 \end{pmatrix}$. Determine the normal strain in the direction of $\hat{\nu} = \hat{e}_1/2 - \hat{e}_2/2 + \hat{e}_3/\sqrt{2}$ and the shear strain between $\hat{\nu}$ and $\hat{\mu} = -\hat{e}_1/2 + \hat{e}_2/2 + \hat{e}_3/\sqrt{2}$. *Ans.* $\epsilon_{(\hat{\nu})} = 6$, $\gamma_{\nu\mu} = 0$.

3.55. Determine the principal-axes form of ϵ_{ij} given in Problem 3.54 and note that $\hat{\nu}$ and $\hat{\mu}$ of that problem are principal directions (hence $\gamma_{\nu\mu} = 0$).

$$Ans.\quad \epsilon_{ij}^* = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

3.56. Draw the Mohr's circle for the state of strain given in Problem 3.54 and determine the maximum shear strain value. Verify this result analytically. *Ans.* $\gamma_{max} = 4$

3.57. Using ϵ_{ij} of Problem 3.54 and ϵ_{ij}^* given in Problem 3.55, calculate the three strain invariants from each and compare the results. *Ans.* $I_E = 6$, $II_E = -4$, $III_E = -24$.

3.58. For ϵ_{ij} of Problem 3.54, determine the deviator tensor e_{ij} and calculate its principal values.

$$Ans.\quad e_{ij} = \begin{pmatrix} -1 & -3 & \sqrt{2} \\ -3 & -1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 2 \end{pmatrix}, \quad e_{ij} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

3.59. A displacement field is given by $u_1 = 3x_1x_2^2$, $u_2 = 2x_3x_1$, $u_3 = x_3^2 - x_1x_2$. Determine the strain tensor ϵ_{ij} and check whether or not the compatibility conditions are satisfied.

$$Ans.\quad \epsilon_{ij} = \begin{bmatrix} 3x_2^2 & 3x_1x_2 + x_3 & -x_2/2 \\ 3x_1x_2 + x_3 & 0 & x_1/2 \\ -x_2/2 & x_1/2 & 2x_3 \end{bmatrix}, \quad \text{Yes.}$$

3.60. For a delta-strain-rosette the normal strains are found to be those shown in Fig. 3-23. Determine ϵ_{12} and ϵ_{22} at the location.
Ans. $\epsilon_{22} = 1 \times 10^{-4}$, $\epsilon_{12} = -0.2885 \times 10^{-4}$

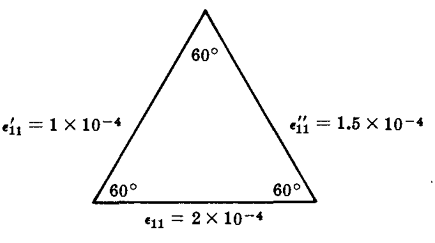


Fig. 3-23

3.61. For the displacement field $x_1 = X_1 + AX_3$, $x_2 = X_2$, $x_3 = X_3 - AX_1$, calculate the volume change and show that it is zero if A is a very small constant.

Chapter 4

Motion and Flow

4.1 MOTION. FLOW. MATERIAL DERIVATIVE

Motion and *flow* are terms used to describe the instantaneous or continuing change in configuration of a continuum. *Flow* sometimes carries the connotation of a motion leading to a permanent deformation as, for example, in plasticity studies. In fluid flow, however, the word denotes continuing motion. As indicated by (*3.14*) and (*3.15*), the motion of a continuum may be expressed either in terms of material coordinates (Lagrangian description) by

$$x_i = x_i(X_1, X_2, X_3, t) = x_i(\mathbf{X}, t) \quad \text{or} \quad \mathbf{x} = \mathbf{x}(\mathbf{X}, t) \tag{4.1}$$

or by the inverse of these equations in terms of the spatial coordinates (Eulerian description) as

$$X_i = X_i(x_1, x_2, x_3, t) = X_i(\mathbf{x}, t) \quad \text{or} \quad \mathbf{X} = \mathbf{X}(\mathbf{x}, t) \tag{4.2}$$

The necessary and sufficient condition for the inverse functions (*4.2*) to exist is that the Jacobian determinant

$$J = |\partial x_i / \partial X_j| \tag{4.3}$$

should not vanish. Physically, the Lagrangian description fixes attention on specific particles of the continuum, whereas the Eulerian description concerns itself with a particular region of the space occupied by the continuum.

Since (*4.1*) and (*4.2*) are the inverses of one another, any physical property of the continuum that is expressed with respect to a specific particle (Lagrangian, or material description) may also be expressed with respect to the particular location in space occupied by the particle (Eulerian, or spatial description). For example, if the material description of the density ρ is given by

$$\rho = \rho(X_i, t) \quad \text{or} \quad \rho = \rho(\mathbf{X}, t) \tag{4.4}$$

the spatial description is obtained by replacing $\mathbf{X}$ in this equation by the function given in (*4.2*). Thus the spatial description of the density is

$$\rho = \rho(X_i(\mathbf{x}, t), t) = \rho^*(x_i, t) \quad \text{or} \quad \rho = \rho(\mathbf{X}(\mathbf{x}, t), t) = \rho^*(\mathbf{x}, t) \tag{4.5}$$

where the symbol ρ^* is used here to emphasize that the functional form of the Eulerian description is not necessarily the same as the Lagrangian form.

The time rate of change of any property of a continuum with respect to specific particles of the moving continuum is called the *material derivative* of that property. The material derivative (also known as the *substantial*, or *comoving*, or *convective* derivative) may be thought of as the time rate of change that would be measured by an observer traveling with the specific particles under study. The instantaneous position x_i of a particle is itself a property of the particle. The material derivative of the particle's position is the *instantaneous velocity* of the particle. Therefore adopting the symbol d/dt or the superpositioned dot ($\cdot$) as representing the operation of material differentiation (some books use D/Dt), the velocity vector is defined by

$$v_i = dx_i/dt = \dot{x}_i \quad \text{or} \quad \mathbf{v} = d\mathbf{x}/dt = \dot{\mathbf{x}} \tag{4.6}$$

In general, if $P_{ij\ldots}$ is any scalar, vector or tensor property of a continuum that may be expressed as a point function of the coordinates, and if the Lagrangian description is given by

$$P_{ij\ldots} = P_{ij\ldots}(\mathbf{X}, t) \tag{4.7}$$

the material derivative of the property is expressed by

$$\frac{dP_{ij\ldots}}{dt} = \frac{\partial P_{ij\ldots}(\mathbf{X}, t)}{\partial t} \tag{4.8}$$

The right-hand side of (*4.8*) is sometimes written $\left[\frac{\partial P_{ij\ldots}(\mathbf{X}, t)}{\partial t}\right]_{\mathbf{X}}$ to emphasize that the $\mathbf{X}$ coordinates are held constant, i.e. the same particles are involved, in taking the derivative. When the property $P_{ij\ldots}$ is expressed by the spatial description in the form

$$P_{ij\ldots} = P_{ij\ldots}(\mathbf{x}, t) \tag{4.9}$$

the material derivative is given by

$$\frac{dP_{ij\ldots}(\mathbf{x}, t)}{dt} = \frac{\partial P_{ij\ldots}(\mathbf{x}, t)}{\partial t} + \frac{\partial P_{ij\ldots}(\mathbf{x}, t)}{\partial x_k}\frac{dx_k}{dt} \tag{4.10}$$

where the second term on the right arises because the specific particles are changing position in space. The first term on the right of (*4.10*) gives the rate of change at a particular location and is accordingly called the *local rate of change*. This term is sometimes written $\left[\frac{\partial P_{ij\ldots}(\mathbf{x}, t)}{\partial t}\right]_{\mathbf{x}}$ to emphasize that $\mathbf{x}$ is held constant in this differentiation. The second term on the right in (*4.10*) is called the *convective rate of change* since it expresses the contribution due to the motion of the particles in the variable field of the property.

From (*4.6*), the material derivative (*4.10*) may be written

$$\frac{dP_{ij\ldots}(\mathbf{x}, t)}{dt} = \frac{\partial P_{ij\ldots}(\mathbf{x}, t)}{\partial t} + v_k\frac{\partial P_{ij\ldots}(\mathbf{x}, t)}{\partial x_k} \tag{4.11}$$

which immediately suggests the introduction of the *material derivative operator*

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_k\frac{\partial}{\partial x_k} \quad \text{or} \quad \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} \tag{4.12}$$

which is used in taking the material derivatives of quantities expressed in spatial coordinates.

4.2 VELOCITY. ACCELERATION. INSTANTANEOUS VELOCITY FIELD

One definition of the velocity vector is given by (*4.6*) as $v_i = dx_i/dt$ (or $\mathbf{v} = d\mathbf{x}/dt$). An alternative definition of the same vector may be obtained from (*3.11*) which gives $x_i = u_i + X_i$ (or $\mathbf{x} = \mathbf{u} + \mathbf{X}$). Thus the velocity may be defined by

$$v_i \equiv \frac{dx_i}{dt} = \frac{d(u_i + X_i)}{dt} = \frac{du_i}{dt} \quad \text{or} \quad \mathbf{v} = \frac{d\mathbf{x}}{dt} = \frac{d(\mathbf{u}+\mathbf{X})}{dt} = \frac{d\mathbf{u}}{dt} \tag{4.13}$$

since $\mathbf{X}$ is independent of time. In (*4.13*), if the displacement is expressed in the Lagrangian form $u_i = u_i(\mathbf{X}, t)$, then

$$v_i \equiv \dot{u}_i = \frac{du_i(\mathbf{X}, t)}{dt} = \frac{\partial u_i(\mathbf{X}, t)}{\partial t} \quad \text{or} \quad \mathbf{v} = \dot{\mathbf{u}} = \frac{d\mathbf{u}(\mathbf{X}, t)}{dt} = \frac{\partial \mathbf{u}(\mathbf{X}, t)}{\partial t} \tag{4.14}$$

If, on the other hand, the displacement is in the Eulerian form $u_i = u_i(\mathbf{x}, t)$, then

$$v_i(\mathbf{x}, t) \equiv \dot{u}_i(\mathbf{x}, t) \equiv \frac{du_i(\mathbf{x}, t)}{dt} = \frac{\partial u_i(\mathbf{x}, t)}{\partial t} + v_k(\mathbf{x}, t)\frac{\partial u_i(\mathbf{x}, t)}{\partial x_k}$$

or

$$\mathbf{v}(\mathbf{x}, t) \equiv \dot{\mathbf{u}}(\mathbf{x}, t) \equiv \frac{d\mathbf{u}(\mathbf{x}, t)}{dt} = \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t} + \mathbf{v}(\mathbf{x}, t) \cdot \nabla_{\mathbf{x}}\, \mathbf{u}(\mathbf{x}, t) \tag{4.15}$$

In (*4.15*) the velocity is given implicitly since it appears as a factor of the second term on the right. The function

$$v_i = v_i(\mathbf{x}, t) \quad \text{or} \quad \mathbf{v} = \mathbf{v}(\mathbf{x}, t) \tag{4.16}$$

is said to specify the *instantaneous velocity field.*

The material derivative of the velocity is the *acceleration.* If the velocity is given in the Lagrangian form (*4.14*), then

$$a_i \equiv \dot{v}_i \equiv \frac{dv_i(\mathbf{X}, t)}{dt} = \frac{\partial v_i(\mathbf{X}, t)}{\partial t} \quad \text{or} \quad \mathbf{a} \equiv \dot{\mathbf{v}} \equiv \frac{d\mathbf{v}(\mathbf{X}, t)}{dt} = \frac{\partial \mathbf{v}(\mathbf{X}, t)}{\partial t} \tag{4.17}$$

If the velocity is given in the Eulerian form (*4.15*), then

$$a_i(x, t) \equiv \frac{dv_i(x, t)}{dt} = \frac{\partial v_i(\mathbf{x}, t)}{\partial t} + v_k(\mathbf{x}, t)\frac{\partial v_i(\mathbf{x}, t)}{\partial x_k}$$

or

$$\mathbf{a}(\mathbf{x}, t) \equiv \frac{d\mathbf{v}(\mathbf{x}, t)}{dt} = \frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial t} + \mathbf{v}(\mathbf{x}, t) \cdot \nabla_{\mathbf{x}}\, \mathbf{v}(\mathbf{x}, t) \tag{4.18}$$

4.3 PATH LINES. STREAM LINES. STEADY MOTION

A *path line* is the curve or path followed by a particle during motion or flow. A *stream line* is the curve whose tangent at any point is in the direction of the velocity at that point. The motion of a continuum is termed *steady motion* if the velocity field is independent of time so that $\partial v_i/\partial t = 0$. For steady motion, stream lines and path lines coincide.

4.4 RATE OF DEFORMATION. VORTICITY. NATURAL STRAIN INCREMENTS

The spatial gradient of the instantaneous velocity field defines the *velocity gradient tensor,* $\partial v_i/\partial x_j$ (or Y_{ij}). This tensor may be decomposed into its symmetric and skew-symmetric parts according to

$$Y_{ij} \equiv \frac{\partial v_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) = D_{ij} + V_{ij}$$

or

$$\mathbf{Y} = \tfrac{1}{2}(\mathbf{v}\nabla_{\mathbf{x}} + \nabla_{\mathbf{x}}\,\mathbf{v}) + \tfrac{1}{2}(\mathbf{v}\nabla_{\mathbf{x}} - \nabla_{\mathbf{x}}\,\mathbf{v}) = (\mathbf{D} + \mathbf{V}) \tag{4.19}$$

This decomposition is valid even if v_i and $\partial v_i/\partial x_j$ are finite quantities. The symmetric tensor

$$D_{ij} = D_{ji} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \quad \text{or} \quad \mathbf{D} = \tfrac{1}{2}(\mathbf{v}\nabla_{\mathbf{x}} + \nabla_{\mathbf{x}}\,\mathbf{v}) \tag{4.20}$$

is called the *rate of deformation tensor.* Many other names are used for this tensor; among them *rate of strain, stretching, strain rate* and *velocity strain* tensor. The skew-symmetric tensor

$$V_{ij} = -V_{ji} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \quad \text{or} \quad \mathbf{V} = \tfrac{1}{2}(\mathbf{v}\nabla_{\mathbf{x}} - \nabla_{\mathbf{x}}\,\mathbf{v}) \tag{4.21}$$

is called the *vorticity* or *spin* tensor.

The rate of deformation tensor is easily shown to be the material derivative of the Eulerian linear strain tensor. Thus if in the equation

$$\frac{d\epsilon_{ij}}{dt} = \frac{1}{2}\frac{d}{dt}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{or} \quad \frac{d\mathbf{E}}{dt} = \frac{1}{2}\frac{d}{dt}(\mathbf{u}\nabla_{\mathbf{x}} + \nabla_{\mathbf{x}}\mathbf{u}) \qquad (4.22)$$

the differentiations with respect to the coordinates and time are interchanged, i.e. if $\frac{d}{dt}\left(\frac{\partial u_i}{\partial x_j}\right)$ is replaced by $\frac{\partial}{\partial x_j}\left(\frac{du_i}{dt}\right)$, the equation takes the form

$$\frac{d\epsilon_{ij}}{dt} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = D_{ij} \quad \text{or} \quad \frac{d\mathbf{E}}{dt} = \tfrac{1}{2}(\mathbf{v}\nabla_{\mathbf{x}} + \nabla_{\mathbf{x}}\mathbf{v}) = \mathbf{D} \qquad (4.23)$$

By the same procedure the vorticity tensor may be shown to be equal to the material derivative of the Eulerian linear rotation tensor. The result is expressed by the equation

$$\frac{d\omega_{ij}}{dt} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) = V_{ij} \quad \text{or} \quad \frac{d\mathbf{\Omega}}{dt} = \tfrac{1}{2}(\mathbf{v}\nabla_{\mathbf{x}} - \nabla_{\mathbf{x}}\mathbf{v}) = \mathbf{V} \qquad (4.24)$$

A rather interesting interpretation may be attached to (*4.23*) when that equation is rewritten in the form

$$d\epsilon_{ij} = D_{ij}\,dt \quad \text{or} \quad d\mathbf{E} = \mathbf{D}\,dt \qquad (4.25)$$

The left hand side of (*4.25*) represents the components known as the *natural strain increments,* widely used in flow problems in the theory of plasticity (see Chapter 8).

4.5 PHYSICAL INTERPRETATION OF RATE OF DEFORMATION AND VORTICITY TENSORS

In Fig. 4-1 the velocities of the neighboring particles at points P and Q in a moving continuum are given by v_i and $v_i + dv_i$ respectively. The relative velocity of the particle at Q with respect to the one at P is therefore

$$dv_i = \frac{\partial v_i}{\partial x_j}dx_j \quad \text{or} \quad d\mathbf{v} = \mathbf{v}\nabla_{\mathbf{x}} \cdot d\mathbf{x} \qquad (4.26)$$

in which the partial derivatives are to be evaluated at P. In terms of D_{ij} and V_{ij}, (*4.26*) becomes

$$dv_i = (D_{ij} + V_{ij})\,dx_j$$

or

$$d\mathbf{v} = (\mathbf{D} + \mathbf{V}) \cdot d\mathbf{x} \qquad (4.27)$$

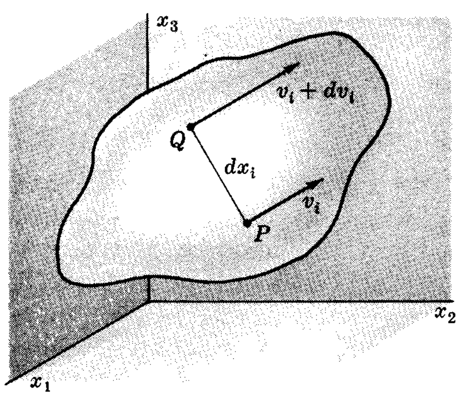


Fig. 4-1

If the rate of deformation tensor is identically zero $(D_{ij} \equiv 0)$,

$$dv_i = V_{ij}\,dx_j \quad \text{or} \quad d\mathbf{v} = \mathbf{V} \cdot d\mathbf{x} \qquad (4.28)$$

and the motion in the neighborhood of P is a *rigid body rotation.* For this reason a velocity field is said to be *irrotational* if the vorticity tensor vanishes everywhere within the field.

Associated with the vorticity tensor, the vector defined by

$$q_i = \epsilon_{ijk}v_{k,j} \quad \text{or} \quad \mathbf{q} = \nabla_{\mathbf{x}} \times \mathbf{v} \qquad (4.29)$$

is known as the *vorticity vector.* The symbolic form of (*4.29*) shows that the vorticity vector is the curl of the velocity field. The vector defined as one-half the vorticity vector,

$$\Omega_i = \tfrac{1}{2}q_i = \tfrac{1}{2}\epsilon_{ijk}v_{k,j} \quad \text{or} \quad \mathbf{\Omega} = \tfrac{1}{2}\mathbf{q} = \tfrac{1}{2}\nabla_{\mathbf{x}} \times \mathbf{v} \tag{4.30}$$

is called the *rate of rotation* vector. For a rigid body rotation such as that described by *(4.28)*, the relative velocity of a neighboring particle separated from P by dx_i is given as

$$dv_i = \epsilon_{ijk}\Omega_j\,dx_k \quad \text{or} \quad d\mathbf{v} = \mathbf{\Omega} \times d\mathbf{x} \tag{4.31}$$

The components of the rate of deformation tensor have the following physical interpretations. The diagonal elements of D_{ij} are known as the *stretching* or *rate of extension* components. Thus for pure deformation, from *(4.27)*,

$$dv_i = D_{ij}\,dx_j \quad \text{or} \quad d\mathbf{v} = \mathbf{D}\cdot d\mathbf{x} \tag{4.32}$$

and, since the rate of change of length of the line element dx_i per unit instantaneous length is given by

$$d_i^{(\nu)} = \frac{dv_i}{dx} = D_{ij}\frac{dx_j}{dx} = D_{ij}\nu_j \quad \text{or} \quad \mathbf{d}^{(\hat{\nu})} = \mathbf{D}\cdot\hat{\boldsymbol{\nu}} \tag{4.33}$$

the *rate of deformation* in the direction of the unit vector ν_i is

$$d = d_i^{(\nu)}\nu_i = D_{ij}\nu_j\nu_i \quad \text{or} \quad d = \hat{\boldsymbol{\nu}}\cdot\mathbf{D}\cdot\hat{\boldsymbol{\nu}} \tag{4.34}$$

From *(4.34)*, if ν_i is in the direction of a coordinate axis, say $\hat{\mathbf{e}}_2$,

$$d = d_{22} \quad \text{or} \quad d = \hat{\mathbf{e}}_2\cdot\mathbf{D}\cdot\hat{\mathbf{e}}_2 = D_{22} \tag{4.35}$$

The off-diagonal elements of D_{ij} are *shear rates,* being a measure of the rate of change between directions at right angles (See Problem **4.18**).

Since D_{ij} is a symmetric, second-order tensor, the concepts of *principal axes, principal values, invariants,* a *rate of deformation quadric,* and a *rate of deformation deviator tensor* may be associated with it. Also, *equations of compatibility* for the components of the rate of deformation tensor, analogous to those presented in Chapter 3 for the linear strain tensors may be developed.

4.6 MATERIAL DERIVATIVES OF VOLUME, AREA AND LINE ELEMENTS

In the motion from some initial configuration at time $t = 0$ to the present configuration at time t, the continuum particles which occupied the differential volume element dV_0 in the initial state now occupy the differential volume element dV. If the initial volume element is taken as the rectangular parallelepiped shown in Fig. 4-2, then by *(1.10)*

$$\begin{aligned} dV &= dX_1\hat{\mathbf{e}}_1 \times dX_2\hat{\mathbf{e}}_2\cdot dX_3\hat{\mathbf{e}}_3 \\ &= dX_1\,dX_2\,dX_3 \end{aligned} \tag{4.36}$$

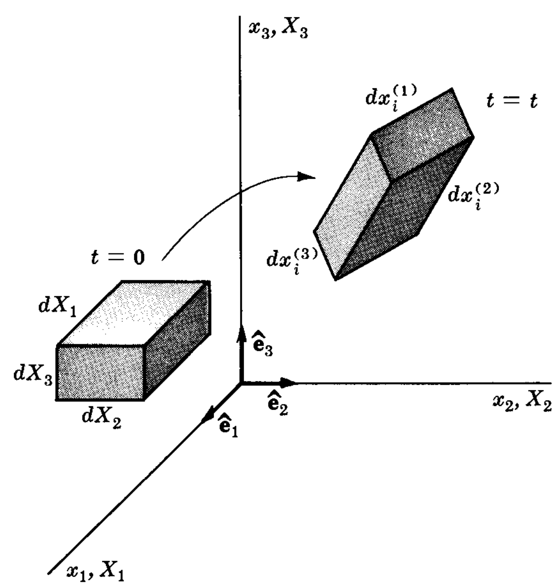


Fig. 4-2

Due to the motion, this parallelepiped is moved and distorted, but because the motion is assumed continuous the volume element does not break up. In fact, because of the relationship *(3.33)*, $dx_i = (\partial x_i/\partial X_j)\,dX_j$ between the material and spatial line elements, the "line of particles" that formed dX_1 now form the differential line segment $dx_i^{(1)} = (\partial x_i/\partial X_1)\,dX_1$. Similarly dX_2 becomes $dx_i^{(2)} = (\partial x_i/\partial X_2)\,dX_2$ and dX_3 becomes $dx_i^{(3)} = (\partial x_i/\partial X_3)\,dX_3$. Therefore the differential volume element dV is a skewed parallelepiped having edges $dx_i^{(1)}, dx_i^{(2)}, dx_i^{(3)}$ and a volume given by the box product

$$dV \;=\; d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)} \cdot d\mathbf{x}^{(3)} \;=\; \epsilon_{ijk}\, dx_i^{(1)}\, dx_j^{(2)}\, dx_k^{(3)} \tag{4.37}$$

But (*4.37*) is seen to be equal to

$$dV \;=\; \epsilon_{ijk} \frac{\partial x_i}{\partial X_1} \frac{\partial x_j}{\partial X_2} \frac{\partial x_k}{\partial X_3}\, dX_1\, dX_2\, dX_3 \;=\; J\, dV_0 \tag{4.38}$$

where $J = |\partial x_i/\partial X_j|$ is the Jacobian defined by (*4.3*).

Using (*4.38*), it is now possible to obtain the material derivative of dV as

$$\frac{d}{dt}(dV) \;=\; \frac{d}{dt}(J\, dV_0) \;=\; \frac{dJ}{dt}\, dV_0 \tag{4.39}$$

since dV_0 is time independent, so that $\frac{d}{dt}(dV_0) = 0$. The material derivative of the Jacobian J may be shown to be (see Problem 4.28)

$$\frac{dJ}{dt} = J \frac{\partial v_i}{\partial x_i} \quad \text{or} \quad \dot{J} \;=\; J \nabla_{\mathbf{x}} \cdot \mathbf{v} \tag{4.40}$$

and hence (*4.39*) may be put into the form

$$\frac{d}{dt}(dV) \;=\; \frac{\partial v_i}{\partial x_i}\, dV \quad \text{or} \quad \frac{d}{dt}(dV) \;=\; \nabla_{\mathbf{x}} \cdot \mathbf{v}\, dV \tag{4.41}$$

For the initial configuration of a continuum, a differential element of area having the magnitude dS^0 may be represented in terms of its unit normal vector n_i by the expression $dS^0\, n_i$. For the current configuration of the continuum in motion, the particles initially making up the area $dS^0\, n_i$ now fill an area element represented by the vector $dS\, n_i$ or dS_i. It may be shown that

$$dS_i \;=\; J \frac{\partial X_j}{\partial x_i}\, dX_j \quad \text{or} \quad d\mathbf{S} \;=\; J\, d\mathbf{X} \cdot \mathbf{X} \nabla_{\mathbf{x}} \tag{4.42}$$

from which the material derivative of the element of area is

$$\frac{d}{dt}(dS_i) \;=\; \frac{d}{dt}\left(J \frac{\partial X_j}{\partial x_i}\right) dX_j \;=\; \frac{\partial v_j}{\partial x_j}\, dS_i - \frac{\partial v_j}{\partial x_i}\, dS_j \tag{4.43}$$

The material derivative of the squared length of the differential line element dx_i may be calculated as follows,

$$\frac{d}{dt}(dx^2) \;=\; \frac{d}{dt}(dx_i\, dx_i) \;=\; 2\, \frac{d(dx_i)}{dt}\, dx_i \tag{4.44}$$

However, since $dx_i = (\partial x_i/\partial X_j)\, dX_j$,

$$\frac{d}{dt}(dx_i) \;=\; \frac{d}{dt}\left(\frac{\partial x_i}{\partial X_j}\right) dX_j \;=\; \frac{\partial v_i}{\partial X_j}\, dX_j \;=\; \frac{\partial v_i}{\partial x_k} \frac{\partial x_k}{\partial X_j}\, dX_j \;=\; \frac{\partial v_i}{\partial x_k}\, dx_k \tag{4.45}$$

and (*4.44*) becomes

$$\frac{d}{dt}(dx^2) \;=\; 2\, \frac{\partial v_i}{\partial x_k}\, dx_k\, dx_i \quad \text{or} \quad \frac{d}{dt}(dx^2) \;=\; 2\, d\mathbf{x} \cdot \nabla_{\mathbf{x}} \mathbf{v} \cdot d\mathbf{x} \tag{4.46}$$

The expression on the right-hand side in the indicial form of (*4.46*) is symmetric in i and k, and accordingly may be written

$$\frac{\partial v_i}{\partial x_k}\, dx_k\, dx_i + \frac{\partial v_k}{\partial x_i}\, dx_i\, dx_k \;=\; \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) dx_i\, dx_k \tag{4.47}$$

or, from (*4.20*),

$$\frac{d}{dt}(dx^2) = 2\,D_{ij}\,dx_i\,dx_j \qquad \text{or} \qquad \frac{d}{dt}(dx^2) = 2\,d\mathbf{x}\cdot\mathbf{D}\cdot d\mathbf{x} \tag{4.48}$$

4.7 MATERIAL DERIVATIVES OF VOLUME, SURFACE AND LINE INTEGRALS

Not all properties of a continuum may be defined for a specific particle as functions of the coordinates such as those given by (*4.7*) and (*4.9*). Some properties are defined as integrals over a finite portion of the continuum. In particular, let any scalar, vector or tensor property be represented by the volume integral

$$P_{ij\ldots}(t) = \int_V P^*_{ij\ldots}(\mathbf{x},t)\,dV \tag{4.49}$$

where V is the volume that the considered part of the continuum occupies at time t. The material derivative of $P_{ij\ldots}(t)$ is

$$\frac{d}{dt}[P_{ij\ldots}(t)] = \frac{d}{dt}\int_V P^*_{ij\ldots}(\mathbf{x},t)\,dV \tag{4.50}$$

and since the differentiation is with respect to a definite portion of the continuum (i.e. a specific mass system), the operations of differentiation and integration may be interchanged. Therefore

$$\frac{d}{dt}\int_V P^*_{ij\ldots}(\mathbf{x},t)\,dV = \int_V \frac{d}{dt}[P^*_{ij\ldots}(\mathbf{x},t)\,dV] \tag{4.51}$$

which, upon carrying out the differentiation and using (*4.41*), results in

$$\frac{d}{dt}\int_V P^*_{ij\ldots}(\mathbf{x},t)\,dV = \int_V \left[\frac{dP^*_{ij\ldots}(\mathbf{x},t)}{dt} + P^*_{ij\ldots}(\mathbf{x},t)\frac{\partial v_p}{\partial x_p}\right]dV \tag{4.52}$$

Since the material derivative operator is given by (*4.12*) as $d/dt = \partial/\partial t + v_p\,\partial/\partial x_p$, (*4.52*) may be put into the form

$$\frac{d}{dt}\int_V P^*_{ij\ldots}(\mathbf{x},t)\,dV = \int_V \left[\frac{\partial P^*_{ij\ldots}(\mathbf{x},t)}{\partial t} + \frac{\partial}{\partial x_p}(v_p P^*_{ij\ldots}(\mathbf{x},t))\right]dV \tag{4.53}$$

By using Gauss' theorem (*1.157*), the second term of the right-hand integral of (*4.53*) may be converted to a surface integral, and the material derivative then given by

$$\frac{d}{dt}\int_V P^*_{ij\ldots}(\mathbf{x},t)\,dV = \int_V \frac{\partial P^*_{ij\ldots}(\mathbf{x},t)}{\partial t}\,dV + \int_S v_p[P^*_{ij\ldots}(\mathbf{x},t)]\,dS_p \tag{4.54}$$

This equation states that the rate of increase of the property $P_{ij\ldots}(t)$ in that portion of the continuum instantaneously occupying V is equal to the sum of the amount of the property created within V plus the flux $v_p[P^*_{ij\ldots}(\mathbf{x},t)]$ through the bounding surface S of V.

The procedure for determining the material derivatives of surface and line integrals is essentially the same as that used above for the volume integral. Thus for any tensorial property of a continuum represented by the surface integral

$$Q_{ij\ldots}(t) = \int_S Q^*_{ij\ldots}(\mathbf{x},t)\,dS_p \tag{4.55}$$

where S is the surface occupied by the considered part of the continuum at time t, then, as before,

$$\frac{d}{dt}\int_S Q^*_{ij\ldots}(\mathbf{x},t)\,dS_p = \int_S \frac{d}{dt}[Q^*_{ij\ldots}(\mathbf{x},t)\,dS_p] \tag{4.56}$$

and, from (4.43), the differentiation in (4.56) yields

$$\frac{dQ_{ij\ldots}(t)}{dt} = \int_S \left[\frac{dQ^*_{ij\ldots}(\mathbf{x},t)}{dt} + \frac{\partial v_q}{\partial x_q} Q^*_{ij\ldots}(\mathbf{x},t)\right] dS_p - \int_S \left[Q^*_{ij\ldots}\frac{\partial v_p}{\partial x_q} dS_p\right] \qquad (4.57)$$

For properties expressed in line integral form such as

$$R_{ij\ldots}(t) = \int_C R^*_{ij\ldots}(\mathbf{x},t)\,dx_p \qquad (4.58)$$

the material derivative is given by

$$\frac{d}{dt}\int_C R^*_{ij\ldots}(\mathbf{x},t)\,dx_p = \int_C \frac{d}{dt}\left[R^*_{ij\ldots}(\mathbf{x},t)\,dx_p\right] \qquad (4.59)$$

Differentiating the right hand integral as indicated in (4.59), and making use of (4.45), results in the material derivative

$$\frac{d}{dt}\left[R_{ij\ldots}(t)\right] = \int_C \frac{d[R^*_{ij\ldots}(\mathbf{x},t)]}{dt}\,dx_p + \int_C \frac{\partial v_p}{\partial x_q}\left[R^*_{ij\ldots}(\mathbf{x},t)\right] dx_q \qquad (4.60)$$

Solved Problems

MATERIAL DERIVATIVES. VELOCITY. ACCELERATION (Sec. 4.1-4.3)

4.1. The spatial (Eulerian) description of a continuum motion is given by $x_1 = X_1 e^t + X_3(e^t - 1)$, $x_2 = X_3(e^t - e^{-t}) + X_2$, $x_3 = X_3$. Show that the Jacobian J does not vanish for this motion and determine the material (Lagrangian) description by inverting the displacement equations.

By (4.3) the Jacobian determinant is

$$J = |\partial x_i/\partial X_j| = \begin{vmatrix} e^t & 0 & e^t - 1 \\ 0 & 1 & e^t - e^{-t} \\ 0 & 0 & 1 \end{vmatrix} = e^t$$

Inverting the motion equations, $X_1 = x_1 e^{-t} + x_3(e^{-t} - 1)$, $X_2 = x_2 - x_3(e^t - e^{-t})$, $X_3 = x_3$. Note that in each description when $t = 0$, $x_i = X_i$.

4.2. A continuum motion is expressed by $x_1 = X_1$, $x_2 = e^t(X_2 + X_3)/2 + e^{-t}(X_2 - X_3)/2$, $x_3 = e^t(X_2 + X_3)/2 - e^{-t}(X_2 - X_3)/2$. Determine the velocity components in both their material and spatial forms.

From the second and third equations, $X_2 + X_3 = e^{-t}(x_2 + x_3)$ and $X_2 - X_3 = e^t(x_2 - x_3)$. Solving these simultaneously the inverse equations become $X_1 = x_1$, $X_2 = e^{-t}(x_2 + x_3)/2 + e^t(x_2 - x_3)/2$, $X_3 = e^{-t}(x_2 + x_3)/2 - e^t(x_2 - x_3)/2$. Accordingly, the displacement components $u_i = x_i - X_i$ may be written in either the Lagrangian form $u_1 = 0$, $u_2 = e^t(X_2 + X_3)/2 + e^{-t}(X_2 - X_3)/2 - X_2$, $u_3 = e^t(X_2 + X_3)/2 - e^{-t}(X_2 - X_3)/2 - X_3$, or in the Eulerian form $u_1 = 0$, $u_2 = x_2 - e^{-t}(x_2 + x_3)/2 - e^t(x_2 - x_3)/2$, $u_3 = x_3 - e^{-t}(x_2 + x_3)/2 + e^t(x_2 - x_3)/2$.

By (4.14), $v_i = \partial u_i/\partial t = \partial X_i/\partial t$ and the velocity components in Lagrangian form are $v_1 = 0$, $v_2 = e^t(X_2 + X_3)/2 - e^{-t}(X_2 - X_3)/2$, $v_3 = e^t(X_2 + X_3)/2 + e^{-t}(X_2 - X_3)/2$. Using the relationships $X_2 + X_3 = e^{-t}(x_2 + x_3)$ and $X_2 - X_3 = e^t(x_2 - x_3)$ these components reduce to $v_1 = 0$, $v_2 = x_3$, $v_3 = x_2$.

Also, from (4.15), for the Eulerian case,

$$du_2/dt = v_2 = e^{-t}(x_2 + x_3)/2 - e^t(x_2 - x_3)/2 + v_2(2 - e^{-t} - e^t)/2 + v_3(-e^{-t} + e^t)/2$$

$$du_3/dt = v_3 = e^{-t}(x_2 + x_3)/2 + e^t(x_2 - x_3)/2 + v_2(-e^{-t} + e^t)/2 + v_3(2 - e^{-t} - e^t)/2$$

Solving these equations simultaneously for v_2 and v_3, the result is as before $v_2 = x_3$, $v_3 = x_2$.

4.3. A velocity field is described by $v_1 = x_1/(1+t)$, $v_2 = 2x_2/(1+t)$, $v_3 = 3x_3/(1+t)$. Determine the acceleration components for this motion.

By (4.18),

$$dv_1/dt = a_1 = -x_1/(1+t)^2 + x_1/(1+t)^2 = 0$$

$$dv_2/dt = a_2 = -2x_2/(1+t)^2 + 4x_2/(1+t)^2 = 2x_2/(1+t)^2$$

$$dv_3/dt = a_3 = -3x_3/(1+t)^2 + 9x_3/(1+t)^2 = 6x_3/(1+t)^2$$

4.4. Integrate the velocity equations of Problem 4.3 to obtain the displacement relations $x_i = x_i(\mathbf{X}, t)$ and from these determine the acceleration components in Lagrangian form for the motion.

By (4.13), $v_1 = dx_1/dt = x_1/(1+t)$; separating variables, $dx_1/x_1 = dt/(1+t)$ which upon integration gives $\ln x_1 = \ln(1+t) + \ln C$ where C is a constant of integration. Since $x_1 = X_1$ when $t = 0$, $C = X_1$ and so $x_1 = X_1(1+t)$. Similar integrations yield $x_2 = X_2(1+t)^2$ and $x_3 = X_3(1+t)^3$.

Thus from (4.14) and (4.17), $v_1 = X_1$, $v_2 = 2X_2(1+t)$, $v_3 = 3X_3(1+t)^2$ and $a_1 = 0$, $a_2 = 2X_2$, $a_3 = 6X_3(1+t)$.

4.5. The motion of a continuum is given by $x_1 = A + (e^{-B\lambda}/\lambda)\sin\lambda(A + \omega t)$, $x_2 = -B - (e^{-B\lambda}/\lambda)\cos\lambda(A + \omega t)$, $x_3 = X_3$. Show that the particle paths are circles and that the velocity magnitude is constant. Also determine the relationship between X_1 and X_2 and the constants A and B.

By writing $x_1 - A = (e^{-B\lambda}/\lambda)\sin\lambda(A + \omega t)$, $x_2 + B = (-e^{-B\lambda}/\lambda)\cos\lambda(A + \omega t)$, then squaring and adding, t is eliminated and the path lines are the circles $(x_1 - A)^2 + (x_2 + B)^2 = e^{-2B\lambda}/\lambda$. From (4.6), $v_1 = \omega e^{-B\lambda}\cos\lambda(A + \omega t)$, $v_2 = \omega e^{-B\lambda}\sin\lambda(A + \omega t)$, $v_3 = 0$ and $v^2 = v_1^2 + v_2^2 + v_3^2 = \omega^2 e^{-2B\lambda}$. Finally, when $t = 0$, $x_i = X_i$ and so $X_1 = A + (e^{-B\lambda}/\lambda)\sin\lambda A$, $X_2 = -B - (e^{-B\lambda}/\lambda)\cos\lambda A$.

4.6. A velocity field is specified by the vector $\mathbf{v} = x_1^2 t\hat{\mathbf{e}}_1 + x_2 t^2\hat{\mathbf{e}}_2 + x_1 x_3 t\hat{\mathbf{e}}_3$. Determine the velocity and acceleration of the particle at $P(1, 3, 2)$ when $t = 1$.

By direct substitution, $\mathbf{v}_P = \hat{\mathbf{e}}_1 + 3\hat{\mathbf{e}}_2 + 2\hat{\mathbf{e}}_3$. Using the vector form of (4.18) the acceleration field is given by

$$\mathbf{a} = x_1^2\hat{\mathbf{e}}_1 + 2x_2 t\hat{\mathbf{e}}_2 + x_1x_3\hat{\mathbf{e}}_3 + (x_1^2 t\hat{\mathbf{e}}_1 + x_2t^2\hat{\mathbf{e}}_2 + x_1x_3t\hat{\mathbf{e}}_3) \cdot (2x_1t\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + x_3t\hat{\mathbf{e}}_1\hat{\mathbf{e}}_3 + t^2\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 + x_1t\hat{\mathbf{e}}_3\hat{\mathbf{e}}_3)$$

or

$$\mathbf{a} = (x_1^2 + 2x_1^3t^2)\hat{\mathbf{e}}_1 + (2x_2t + x_2t^4)\hat{\mathbf{e}}_2 + (x_1x_3 + 2x\ \ x_3t^2)\hat{\mathbf{e}}_3$$

Thus $\mathbf{a}_P = 3\hat{\mathbf{e}}_1 + 9\hat{\mathbf{e}}_2 + 6\hat{\mathbf{e}}_3$.

4.7. For the velocity field of Problem 4.3 determine the streamlines and path lines of the flow and show that they coincide.

At every point on a streamline the tangent is in the direction of the velocity. Hence for the differential tangent vector $d\mathbf{x}$ along the streamline, $\mathbf{v} \times d\mathbf{x} = \mathbf{0}$ and accordingly the differential equations of the streamlines become $dx_1/v_1 = dx_2/v_2 = dx_3/v_3$. For the given flow these equations are $dx_1/x_1 = dx_2/2x_2 = dx_3/3x_3$. Integrating and using the conditions $x_i = X_i$ when $t = 0$, the equations of the streamlines are $(x_1/X_1)^2 = x_2/X_2$, $(x_1/X_1)^3 = x_3/X_3$, $(x_2/X_2)^3 = (x_3/X_3)^2$.

Integration of the velocity expressions $dx_i/dt = v_i$ as was carried out in Problem 4.4 yields the displacement equations $x_1 = X_1(1+t)$, $x_2 = X_2(1+t)^2$, $x_3 = X_3(1+t)^3$. Eliminating t from these equations gives the path lines which are identical with the streamlines presented above.

4.8. The magnetic field strength of an electromagnetic continuum is given by $\lambda = e^{-At}/r$ where $r^2 = x_1^2 + x_2^2 + x_3^2$ and A is a constant. If the velocity field of the continuum is given by $v_1 = Bx_1x_3t$, $v_2 = Bx_2^2t^2$, $v_3 = Bx_3x_2$, determine the rate of change of magnetic intensity for the particle at $P(2,-1,2)$ when $t = 1$.

Since $\partial(r^{-1})/\partial x_i = -x_i/r^3$, equation (4.11) gives

$$\dot{\lambda} = -Ae^{-At}/r - e^{-At}(Bx_1^2x_3t + Bx_2^3t^2 + Bx_3^2x_2)/r^3$$

Thus for P at $t = 1$, $\dot{\lambda}_P = -e^{-A}(3A + B)/9$.

4.9. A velocity field is given by $v_1 = 4x_3 - 3x_2$, $v_2 = 3x_1$, $v_3 = -4x_1$. Determine the acceleration components at $P(b,0,0)$ and $Q(0,4b,3b)$ and note that the velocity field corresponds to a rigid body rotation of angular velocity 5 about the axis along $\hat{\mathbf{e}} = (4\hat{\mathbf{e}}_2 + 3\hat{\mathbf{e}}_3)/5$.

From (4.18), $a_1 = -25x_1$, $a_2 = -9x_2 + 12x_3$, $a_3 = 12x_2 - 16x_3$. Thus at $P(b,0,0)$, $\mathbf{a} = -25b\,\hat{\mathbf{e}}_1$ which is a normal component of acceleration. Also, at $Q(0,4b,3b)$ which is on the axis of rotation, $\mathbf{a} = 0$. Note that $\mathbf{v} = \mathbf{w} \times \mathbf{x} = (4\,\hat{\mathbf{e}}_2 + 3\,\hat{\mathbf{e}}_3) \times (x_1\,\hat{\mathbf{e}}_1 + x_2\,\hat{\mathbf{e}}_2 + x_3\,\hat{\mathbf{e}}_3) = (4x_3 - 3x_2)\,\hat{\mathbf{e}}_1 + 3x_1\,\hat{\mathbf{e}}_2 - 4x_1\,\hat{\mathbf{e}}_3$.

RATE OF DEFORMATION, VORTICITY (Sec. 4.4-4.5)

4.10. A certain flow is given by $v_1 = 0$, $v_2 = A(x_1x_2 - x_3^2)e^{-Bt}$, $v_3 = A(x_2^2 - x_1x_3)e^{-Bt}$ where A and B are constants. Determine the velocity gradient $\partial v_i/\partial x_j$ for this motion and from it compute the rate of deformation tensor **D** and the spin tensor **V** for the point $P(1,0,3)$ when $t = 0$.

By (4.19), $\partial v_i/\partial x_j = \begin{pmatrix} 0 & 0 & 0 \\ x_2 & x_1 & -2x_3 \\ -x_3 & 2x_2 & -x_1 \end{pmatrix} Ae^{-Bt}$ which may be evaluated at P when $t = 0$

and decomposed according to (4.20) and (4.21) as

$$\mathbf{Y} = \mathbf{D} + \mathbf{V} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A & -6A \\ -3A & 0 & -A \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1.5A \\ 0 & A & -3A \\ -1.5A & -3A & -A \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1.5A \\ 0 & 0 & -3A \\ -1.5A & 3A & 0 \end{pmatrix}$$

4.11. For the motion $x_1 = X_1$, $x_2 = X_2 + X_1(e^{-2t} - 1)$, $x_3 = X_3 + X_1(e^{-3t} - 1)$ compute the rate of deformation **D** and the vorticity tensor **V**. Compare **D** with $d\epsilon_{ij}/dt$, the rate of change of the Eulerian small strain tensor **E**.

Here the displacement components are $u_1 = 0$, $u_2 = x_1(e^{-2t} - 1)$, $u_3 = x_1(e^{-3t} - 1)$ and from (4.14) the velocity components are $v_1 = 0$, $v_2 = -2x_1e^{-2t}$, $v_3 = -3x_1e^{-3t}$. Decomposition of the velocity gradient $\partial v_i/\partial x_j$ gives $\partial v_i/\partial x_j = D_{ij} + V_{ij}$. Thus

$$\partial v_i/\partial x_j = \begin{pmatrix} 0 & 0 & 0 \\ -2e^{-2t} & 0 & 0 \\ -3e^{-3t} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -e^{-2t} & -3e^{-3t}/2 \\ -e^{-2t} & 0 & 0 \\ -3e^{-3t}/2 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & e^{-2t} & 3e^{-3t}/2 \\ -e^{-2t} & 0 & 0 \\ -3e^{-3t}/2 & 0 & 0 \end{pmatrix}$$

Likewise, decomposition of the displacement gradient gives $\partial u_i/\partial x_j = \epsilon_{ij} + \omega_{ij}$. Thus

$$\partial u_i/\partial x_j = \begin{pmatrix} 0 & 0 & 0 \\ e^{-2t} & 0 & 0 \\ e^{-3t} & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & e^{-2t} & e^{-3t} \\ e^{-2t} & 0 & 0 \\ e^{-3t} & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & -e^{-2t} & -e^{-3t} \\ e^{-2t} & 0 & 0 \\ e^{-3t} & 0 & 0 \end{pmatrix}$$

Comparing **D** with $d\mathbf{E}/dt$,

$$d\epsilon_{ij}/dt = \begin{pmatrix} 0 & -e^{-2t} & -3e^{-3t}/2 \\ -e^{-2t} & 0 & 0 \\ -3e^{-3t}/2 & 0 & 0 \end{pmatrix} = D_{ij}$$

The student should show that $d\omega_{ij}/dt = V_{ij}$.

4.12. A vortex line is one whose tangent at every point in a moving continuum is in the direction of the vorticity vector $\mathbf{q}$. Show that the equations for vortex lines are $dx_1/q_1 = dx_2/q_2 = dx_3/q_3$.

Let $d\mathbf{x}$ be a differential distance vector in the direction of $\mathbf{q}$. Then $\mathbf{q} \times d\mathbf{x} \equiv \mathbf{0}$, or

$$(q_2\,dx_3 - q_3\,dx_2)\,\hat{\mathbf{e}}_1 + (q_3\,dx_1 - q_1\,dx_3)\,\hat{\mathbf{e}}_2 + (q_1\,dx_2 - q_2\,dx_1)\,\hat{\mathbf{e}}_3 \equiv 0$$

from which $dx_1/q_1 = dx_2/q_2 = dx_3/q_3$.

4.13. Show that for the velocity field $\mathbf{v} = (Ax_3 - Bx_2)\hat{\mathbf{e}}_1 + (Bx_1 - Cx_3)\hat{\mathbf{e}}_2 + (Cx_2 - Ax_1)\hat{\mathbf{e}}_3$ the vortex lines are straight lines and determine their equations.

From (*4.29*), $\mathbf{q} = \nabla_{\mathbf{x}} \times \mathbf{v} = 2(C\,\hat{\mathbf{e}}_1 + A\,\hat{\mathbf{e}}_2 + B\,\hat{\mathbf{e}}_3)$, and by Problem 4.12, the d.e. for the vortex lines are $A\,dx_3 = B\,dx_2$, $B\,dx_1 = C\,dx_3$, $C\,dx_2 = A\,dx_1$. Integrating these in turn yields the equations of the vortex lines $x_3 = Bx_2/A + K_1$, $x_1 = Cx_3/B + K_2$, $x_2 = Ax_1/C + K_3$ where the K_i are constants of integration.

4.14. Show that the velocity field of Problem 4.13 represents a rigid body rotation by showing that $\mathbf{D} \equiv 0$.

Calculating the velocity gradient $\partial v_i/\partial x_j$, it is found to be antisymmetric. Thus $\partial v_i/\partial x_j =$

$$\begin{pmatrix} 0 & -B & A \\ B & 0 & -C \\ -A & C & 0 \end{pmatrix} = V_{ij} \text{ and } D_{ij} \equiv 0.$$

4.15. For the rigid body rotation $\mathbf{v} = 3x_3\hat{\mathbf{e}}_1 - 4x_3\hat{\mathbf{e}}_2 + (4x_2 - 3x_1)\hat{\mathbf{e}}_3$, determine the rate of rotation vector $\mathbf{\Omega}$ and show that $\mathbf{v} = \mathbf{\Omega} \times \mathbf{x}$.

From (*4.30*), $2\mathbf{\Omega} = \mathbf{q}$, or $\mathbf{\Omega} = 4\,\hat{\mathbf{e}}_1 + 3\,\hat{\mathbf{e}}_2$. This vector is along the axis of rotation. Thus

$$(4\,\hat{\mathbf{e}}_1 + 3\,\hat{\mathbf{e}}_2) \times (x_1\,\hat{\mathbf{e}}_1 + x_2\,\hat{\mathbf{e}}_2 + x_3\,\hat{\mathbf{e}}_3) = 3x_3\,\hat{\mathbf{e}}_1 - 4x_3\,\hat{\mathbf{e}}_2 + (4x_2 - 3x_1)\,\hat{\mathbf{e}}_3 = \mathbf{v}$$

4.16. A steady velocity field is given by $\mathbf{v} = (x_1^3 - x_1x_2^2)\hat{\mathbf{e}}_1 + (x_1^2x_2 + x_2)\hat{\mathbf{e}}_2$. Determine the unit relative velocity with respect to $P(1,1,3)$ of the particles at $Q_1(1,0,3)$, $Q_2(1,3/4,3)$, $Q_3(1,7/8,3)$ and show that these values approach the relative velocity given by (*4.26*).

By direct calculation $\mathbf{v}_P - \mathbf{v}_{Q_1} = -\hat{\mathbf{e}}_1 + 2\,\hat{\mathbf{e}}_2$, $4(\mathbf{v}_P - \mathbf{v}_{Q_2}) = -7\,\hat{\mathbf{e}}_1/4 + 2\,\hat{\mathbf{e}}_2$ and $8(\mathbf{v}_P - \mathbf{v}_{Q_3}) = -15\,\hat{\mathbf{e}}_1/8 + 2\,\hat{\mathbf{e}}_2$. The velocity gradient matrix is

$$[\partial v_i/\partial x_j] = \begin{bmatrix} 3x_1^2 - x_2^2 & -2x_1x_2 & 0 \\ 2x_1x_2 & x_1^2 + 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and at $P(1,1,3)$ in the negative x_2 direction,

$$(dv_i/dx)_{\hat{\mathbf{e}}_2} = \begin{bmatrix} 2 & -2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} +2 \\ -2 \\ 0 \end{bmatrix}$$

Thus $(d\mathbf{v}/dx)_{\hat{\mathbf{e}}_2} = -2\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2$ which is the value approached by the relative unit velocities $\mathbf{v}_P - \mathbf{v}_{Q_i}$.

4.17. For the steady velocity field $\mathbf{v} = 3x_1^2x_2\hat{\mathbf{e}}_1 + 2x_2^2x_3\hat{\mathbf{e}}_2 + x_1x_2x_3^2\hat{\mathbf{e}}_3$, determine the rate of extension at $P(1,1,1)$ in the direction of $\hat{\boldsymbol{\nu}} = (3\hat{\mathbf{e}}_1 - 4\hat{\mathbf{e}}_3)/5$.

Here the velocity gradient is $[\partial v_i/\partial x_j] = \begin{bmatrix} 6x_1x_2 & 3x_1^2 & 0 \\ 0 & 4x_2x_3 & 2x_2^2 \\ x_2x_3^2 & x_1x_3^2 & 2x_1x_2x_3 \end{bmatrix}$ and its symmetric part at P is $[D_{ij}] = \begin{bmatrix} 6 & 1.5 & 0.5 \\ 1.5 & 4 & 1.5 \\ 0.5 & 1.5 & 2 \end{bmatrix}$.

Thus from (*4.34*) for $\hat{\boldsymbol{\nu}} = (3\hat{\mathbf{e}}_1 - 4\hat{\mathbf{e}}_3)/5$,

$$d = [3/5, 0, -4/5] \begin{bmatrix} 6 & 1.5 & 0.5 \\ 1.5 & 4 & 1.5 \\ 0.5 & 1.5 & 2 \end{bmatrix} \begin{bmatrix} 3/5 \\ 0 \\ -4/5 \end{bmatrix} = 74/25$$

4.18. For the motion of Problem 4.17 determine the rate of shear at P between the orthogonal directions $\hat{\boldsymbol{\nu}} = (3\hat{\mathbf{e}}_1 - 4\hat{\mathbf{e}}_3)/5$ and $\hat{\boldsymbol{\mu}} = (4\hat{\mathbf{e}}_1 + 3\hat{\mathbf{e}}_3)/5$.

In analogy with the results of Problem 3.20 the shear rate $\dot{\gamma}_{\mu\nu}$ is given by $\dot{\gamma}_{\mu\nu} = \hat{\boldsymbol{\mu}} \cdot 2\mathbf{D} \cdot \hat{\boldsymbol{\nu}}$, or in matrix form

$$\dot{\gamma}_{\mu\nu} = [4/5, 0, 3/5] \begin{bmatrix} 12 & 3 & 1 \\ 3 & 8 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 3/5 \\ 0 \\ -4/5 \end{bmatrix} = 89/25$$

4.19. A steady velocity field is given by $v_1 = 2x_3$, $v_2 = 2x_3$, $v_3 = 0$. Determine the principal directions and principal values (rates of extension) of the rate of deformation tensor for this motion.

Here $[\partial v_i/\partial x_j] = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ and for principal values λ of D_{ij},

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0 = -\lambda^3 + 2\lambda$$

Thus $\lambda_{\mathrm{I}} = +\sqrt{2}$, $\lambda_{\mathrm{II}} = 0$, $\lambda_{\mathrm{III}} = -\sqrt{2}$.

The transformation matrix to principal axis directions is

$$[a_{ij}] = \begin{bmatrix} -1/2 & -1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/2 & 1/2 & 1/\sqrt{2} \end{bmatrix}$$

with the rate of deformation matrix in principal form $[D^*_{ij}] = \begin{bmatrix} +\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} \end{bmatrix}$.

4.20. Determine the maximum shear rate $\dot{\gamma}_{\max}$ for the motion of Problem 4.19.

Analogous to principal shear strains of Chapter 3, the maximum shear rate is $\dot{\gamma}_{\max} = (\lambda_I - \lambda_{III})/2 = \sqrt{2}$.

This result is also available by observing that the motion is a simple shearing parallel to the x_1x_2 plane in the direction of the unit vector $\hat{\nu} = (\hat{e}_1 + \hat{e}_2)/\sqrt{2}$. Thus, as before,

$$\dot{\gamma}_{\max} = \dot{\gamma}_{\mu\nu} = [0,0,1]\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \sqrt{2}$$

It is also worth noting that the maximum rate of extension for this motion occurs in the direction $\hat{n} = (\hat{e}_1 + \hat{e}_2 + \sqrt{2}\,\hat{e}_3)/2$ as found in Problem 4.19. Thus

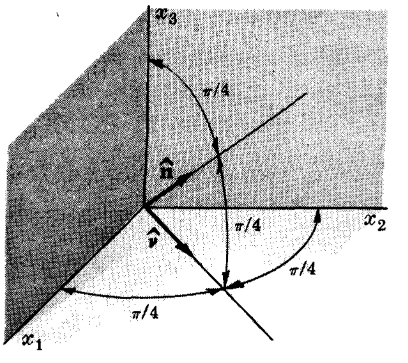


Fig. 4-3

$$\lambda_I = d^{(\hat{n})} = [1/2,\ 1/2,\ \sqrt{2}/2]\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1/2 \\ 1/2 \\ \sqrt{2}/2 \end{bmatrix} = \sqrt{2}$$

MATERIAL DERIVATIVES OF VOLUMES, AREAS, INTEGRALS, ETC. (Sec. 4.6-4.7)

4.21. Calculate the second material derivative of the scalar product of two line elements, i.e. determine $d^2(dx^2)/dt^2$.

From (*4.45*), $\dfrac{d(dx_i)}{dt} = \dfrac{\partial v_i}{\partial x_k}\,dx_k$; and it is shown in (*4.48*) that $\dfrac{d(dx^2)}{dt} = 2D_{ij}\,dx_i\,dx_j$. Therefore

$$\frac{d^2(dx^2)}{dt^2} = 2\left[\frac{dD_{ij}}{dt}\,dx_i\,dx_j + D_{ij}\frac{\partial v_i}{\partial x_k}\,dx_k\,dx_j + D_{ij}\,dx_i\frac{\partial v_j}{\partial x_k}\,dx_k\right]$$

and by simple manipulation of the dummy indices,

$$\frac{d^2(dx^2)}{dt^2} = 2\left[\frac{dD_{ij}}{dt} + D_{kj}\frac{\partial v_k}{\partial x_i} + D_{ik}\frac{\partial v_k}{\partial x_j}\right]dx_i\,dx_j$$

4.22. Determine the material derivative $\dfrac{d}{dt}\displaystyle\int_S p_i\,dS_i$ of the flux of the vector property p_i through the surface S.

By (*4.57*),

$$\frac{d}{dt}\int_S p_i\,dS_i = \int_S\left[\frac{dp_i}{dt} + p_i\frac{\partial v_k}{\partial x_k}\right]dS_i - \int_S p_i\frac{\partial v_k}{\partial x_i}\,dS_k = \int_S\left[\frac{dp_i}{dt} + p_i\frac{\partial v_k}{\partial x_k} - p_k\frac{\partial v_i}{\partial x_k}\right]dS_k$$

4.23. Show that the transport theorem derived in Problem 4.22 may be written in symbolic notation as

$$\frac{d}{dt}\int_S \mathbf{p}\cdot\hat{\mathbf{n}}\,dS = \int_S\left[\frac{\partial\mathbf{p}}{\partial t} + \mathbf{v}(\nabla\cdot\mathbf{p}) + \nabla\times(\mathbf{p}\times\mathbf{v})\right]\cdot\hat{\mathbf{n}}\,dS$$

By a direct transcription into symbolic notation of the result in Problem 4.22,

$$\frac{d}{dt}\int_S \mathbf{p}\cdot\hat{\mathbf{n}}\,dS = \int_S\left[\frac{d\mathbf{p}}{dt} + \mathbf{p}(\nabla\cdot\mathbf{v}) - (\mathbf{p}\cdot\nabla)\mathbf{v}\right]\cdot\hat{\mathbf{n}}\,dS$$

$$= \int_S\left[\frac{\partial\mathbf{p}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{p} + \mathbf{p}(\nabla\cdot\mathbf{v}) - (\mathbf{p}\cdot\nabla)\mathbf{v}\right]\cdot\hat{\mathbf{n}}\,dS$$

Now use of the vector identity $\nabla \times (\mathbf{p} \times \mathbf{v}) = \mathbf{p}(\nabla \cdot \mathbf{v}) - \mathbf{v}(\nabla \cdot \mathbf{p}) + (\mathbf{v} \cdot \nabla)\mathbf{p} - (\mathbf{p} \cdot \nabla)\mathbf{v}$ (see Problem 1.65) gives

$$\frac{d}{dt}\int_S \mathbf{p} \cdot \hat{\mathbf{n}}\, dS \;=\; \int_S \left[\frac{\partial \mathbf{p}}{\partial t} + \mathbf{v}(\nabla \cdot \mathbf{p}) + \nabla \times (\mathbf{p} \times \mathbf{v})\right] \cdot \hat{\mathbf{n}}\, dS$$

4.24. Express *Reynold's transport theorem* as given by equations (*4.53*) and (*4.54*) in symbolic notation.

Let $P^*(\mathbf{x}, t)$ be any tensor function of the Eulerian coordinates and time. Then (*4.53*) is

$$\frac{d}{dt}\int_V P^*(\mathbf{x}, t)\, dV \;=\; \int_V \left[\frac{\partial P^*}{\partial t} + \nabla \cdot (P^* \mathbf{v})\right] dV$$

and by Gauss' divergence theorem this becomes (*4.54*),

$$\frac{d}{dt}\int_V P^*(\mathbf{x}, t)\, dV \;=\; \int_V \frac{\partial P^*}{\partial t}\, dV + \int_S P^* \mathbf{v} \cdot \hat{\mathbf{n}}\, dS$$

4.25. If the function $P^*(\mathbf{x}, t)$ in Problem **4.24** is the scalar 1, the integral on the left is simply the instantaneous volume of a portion of the continuum. Determine the material derivative of this volume.

Using the vector form of (*4.53*) as given in Problem 4.24, $\frac{d}{dt}\int_V dV = \int_V \nabla \cdot \mathbf{v}\, dV$. Here $\nabla \cdot \mathbf{v}\, dV$ represents the rate of change of dV, and so $\nabla \cdot \mathbf{v}$ is known as the cubical rate of dilatation. This relationship may also be established by a direct differentiation of (*4.38*). See Problem 4.43.

MISCELLANEOUS PROBLEMS

4.26. From the definition of the vorticity vector (*4.29*), $\mathbf{q} = \operatorname{curl} \mathbf{v}$, show that $q_i = \epsilon_{ijk} V_{kj}$ and that $2V_{ij} = \epsilon_{jik} q_k$.

By (*4.29*), $q_i = \epsilon_{ijk} v_{k,j} = \epsilon_{ijk}(v_{[k,j]} + v_{(k,j)})$ and since $\epsilon_{ijk} v_{(k,j)} = 0$ (see, for example, Problem 1.50), $q_i = \epsilon_{ijk} v_{[k,j]} = \epsilon_{ijk} V_{kj}$. From this result $\epsilon_{irs} q_i = \epsilon_{irs}\epsilon_{ijk} V_{kj} = (\delta_{rj}\delta_{sk} - \delta_{rk}\delta_{sj}) V_{kj} = 2V_{sr}$.

4.27. Show that the acceleration **a** may be written as $\mathbf{a} = \dfrac{\partial \mathbf{v}}{\partial t} + \mathbf{q} \times \mathbf{v} + \frac{1}{2}\nabla v^2$.

From (*4.18*), $a_i = \dfrac{\partial v_i}{\partial t} + v_k \dfrac{\partial v_i}{\partial x_k}$ and so

$$\begin{aligned} a_i &= \frac{\partial v_i}{\partial t} + v_k\left(\frac{\partial v_i}{\partial x_k} - \frac{\partial v_k}{\partial x_i}\right) + v_k \frac{\partial v_k}{\partial x_i} \\ &= \frac{\partial v_i}{\partial t} + 2 v_k V_{ik} + \frac{1}{2}\frac{\partial (v_k v_k)}{\partial x_i} \;=\; \frac{\partial v_i}{\partial t} + \epsilon_{ijk} q_j v_k + \frac{1}{2}\frac{\partial (v_k v_k)}{\partial x_i} \end{aligned}$$

which, as the student should confirm, is the indicial form of the required equation.

4.28. Show that $d(\ln J)/dt = \operatorname{div} \mathbf{v}$.

Let $\partial x_i / \partial X_p$ be written here as $x_{i,P}$ so that $J = \epsilon_{PQR}\, x_{1,P} x_{2,Q} x_{3,R}$ and $\dot{J}$ becomes the sum of the three determinants, $\dot{J} = \epsilon_{PQR}(\dot{x}_{1,P} x_{2,Q} x_{3,R} + x_{1,P} \dot{x}_{2,Q} x_{3,R} + x_{1,P} x_{2,Q} \dot{x}_{3,R})$. Now $\dot{x}_{1,P} = v_{1,s} x_{s,P}$, etc., and so $\dot{J} = \epsilon_{PQR}(v_{1,s} x_{s,P} x_{2,Q} x_{3,R} + x_{1,P} v_{2,s} x_{s,Q} x_{3,R} + x_{1,P} x_{2,Q} v_{3,s} x_{s,R})$. Of the nine 3×3 determinants resulting from summation on s in this expression, the three non-vanishing ones yield $\dot{J} = v_{1,1} J + v_{2,2} J + v_{3,3} J = v_{s,s} J$. Thus $\dot{J} = J \nabla \cdot \mathbf{v}$ and so $d(\ln J)/dt = \operatorname{div} \mathbf{v}$.

4.29. Show that for steady motion $(\partial v_i/\partial t = 0)$ of a continuum the streamlines and pathlines coincide.

As shown in Problem 4.7, at a given instant t streamlines are the solutions of the differential equations $dx_1/v_1 = dx_2/v_2 = dx_3/v_3$. Pathlines are solutions of the differential equations $dx_i/dt = v_i(\mathbf{x}, t)$. If $v_i = v_i(\mathbf{x})$, these equations become $dt = dx_1/v_1 = dx_2/v_2 = dx_3/v_3$ which coincide with the streamline differential equations.

4.30. For the steady velocity field $v_1 = x_1^2 x_2 + x_2^3$, $v_2 = -x_1^3 - x_1 x_2^2$, $v_3 = 0$ determine expressions for the principal values of the rate of deformation tensor $\mathbf{D}$ at an arbitrary point $P(x_1, x_2, x_3)$.

By (4.19) $\partial v_i/\partial x_j = D_{ij} + V_{ij}$, or

$$\begin{pmatrix} 2x_1x_2 & x_1^2+3x_2^2 & 0 \\ -3x_1^2 - x_2^2 & -2x_1x_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2x_1x_2 & -x_1^2+x_2^2 & 0 \\ -x_1^2+x_2^2 & -2x_1x_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2(x_1^2+x_2^2) & 0 \\ -2(x_1^2+x_2^2) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Principal values $d_{(i)}$ are solutions of

$$\begin{vmatrix} 2x_1x_2 - d & -x_1^2+x_2^2 & 0 \\ -x_1^2+x_2^2 & -2x_1x_2-d & 0 \\ 0 & 0 & -d \end{vmatrix} = 0 = -d[-4x_1^2x_2^2 + d^2 - (x_2^2 - x_1^2)^2]$$

Thus $d_{(1)} = 0$, $d_{(2)} = -(x_1^2 + x_2^2)$, $d_{(3)} = (x_1^2 + x_2^2)$. Note here that $d_{\mathrm{I}} = (x_1^2 + x_2^2)$, $d_{\mathrm{II}} = 0$, $d_{\mathrm{III}} = -(x_1^2 + x_2^2)$.

4.31. Prove equation (4.43) by taking the material derivative of dS_i in its cross product form $dS_i = \epsilon_{ijk}\, dx_j^{(2)}\, dx_k^{(3)}$.

Using (3.33), $dS_i = \epsilon_{ijk}(\partial x_j/\partial X_2)\, dX_2(\partial x_k/\partial X_3)\, dX_3$ and $\dfrac{\partial x_i}{\partial X_i}\, dS_i = \epsilon_{ijk} \dfrac{\partial x_i}{\partial X_1} \dfrac{\partial x_j}{\partial X_2} \dfrac{\partial x_k}{\partial X_3}\, dX_2\, dX_3 = J\, dX_2\, dX_3$. Thus $\dfrac{\partial X_1}{\partial x_p}\dfrac{\partial x_i}{\partial X_1}\, dS_i = \delta_{ip}\, dS_i = dS_p = \dfrac{\partial X_1}{\partial x_p} J\, dX_2\, dX_3$ and by Problem 4.28,

$$\begin{aligned} \frac{dS_p}{dt} &= \left(\frac{\partial X_1}{\partial x_p} J \frac{\partial v_q}{\partial x_q} - J \frac{\partial X_1}{\partial x_q}\frac{\partial v_q}{\partial x_p}\right) dX_2\, dX_3 \\ &= \left(\epsilon_{pjk} \frac{\partial x_j}{\partial X_2}\, dX_2 \frac{\partial x_k}{\partial X_3}\, dX_3\right)\frac{\partial v_q}{\partial x_q} - \left(\epsilon_{qjk}\frac{\partial x_j}{\partial X_2}\, dX_2 \frac{\partial x_k}{\partial X_3}\, dX_3\right)\frac{\partial v_q}{\partial x_p} \\ &= (\partial v_q/\partial x_q)\, dS_p - (\partial v_q/\partial x_p)\, dS_q \end{aligned}$$

4.32. Use the results of Problems 4.27 and 4.23 to show that the material rate of change of the vorticity flux $\dfrac{d}{dt}\displaystyle\int_S \mathbf{q}\cdot\hat{\mathbf{n}}\, dS$ equals the flux of the curl of the acceleration $\mathbf{a}$.

Taking the curl of the acceleration as given in Problem 4.27,

$$\nabla \times \mathbf{a} = \nabla \times \frac{\partial \mathbf{v}}{\partial t} + \nabla \times (\mathbf{q} \times \mathbf{v}) + \nabla \times \nabla(v^2/2)$$

or

$$\nabla \times \mathbf{a} = \partial \mathbf{q}/\partial t + \nabla \times (\mathbf{q}\times\mathbf{v}) = d\mathbf{q}/dt + \mathbf{q}(\nabla\cdot\mathbf{v}) - (\mathbf{q}\cdot\nabla)\mathbf{v}$$

since $\mathbf{q} = \nabla \times \mathbf{v}$ and $\nabla \times \nabla(v^2/2) = 0$. Thus if $\mathbf{q}$ is substituted for $\mathbf{p}$ in Problem 4.23,

$$\frac{d}{dt}\int_S \mathbf{q}\cdot\hat{\mathbf{n}}\, dS = \int_S \left[\frac{d\mathbf{q}}{dt} + \mathbf{q}(\nabla\cdot\mathbf{v}) - (\mathbf{q}\cdot\nabla)\mathbf{v}\right]\cdot\hat{\mathbf{n}}\, dS = \int_S (\nabla\times\mathbf{a})\cdot\hat{\mathbf{n}}\, dS$$

4.33. For the vorticity q_i show that $\frac{\partial}{\partial t}\int_V q_i\,dV = \int_S [\epsilon_{ijk}a_k + q_jv_i - q_iv_j]\,dS_j$.

From Problem 4.32 the identity $\nabla \times \mathbf{a} = \partial\mathbf{q}/\partial t + \nabla \times (\mathbf{q}\times\mathbf{v})$ may be written in indicial form as $\partial q_i/\partial t = \epsilon_{ijk}a_{k,j} - \epsilon_{isp}(\epsilon_{pmr}q_mv_r)_{,s}$. Thus

$$\int_V \frac{\partial q_i}{\partial t}dV = \int_V [\epsilon_{ijk}a_{k,j} - (\epsilon_{isp}\epsilon_{pmr}q_mv_r)_{,s}]\,dV$$

and by the divergence theorem of Gauss (1.157),

$$\int_V \frac{\partial q_i}{\partial t}dV = \int_S \epsilon_{ijk}a_k\,dS_j - \int_S (\delta_{im}\delta_{sr} - \delta_{ir}\delta_{sm})(q_mv_r)\,dS_s = \int_S [\epsilon_{ijk}a_k + q_jv_i - q_iv_j]\,dS_j$$

Supplementary Problems

4.34. A continuum motion is given by $x_1 = X_1e^t + X_3(e^t-1)$, $x_2 = X_2 + X_3(e^t - e^{-t})$, $x_3 = X_3$. Show that J does not vanish for this motion and obtain the velocity components.
Ans. $v_1 = (X_1+X_3)e^t$, $v_2 = X_3(e^t+e^{-t})$, $v_3 = 0$ or $v_1 = x_1 - x_3$, $v_2 = x_3(e^t+e^{-t})$, $v_3 = 0$

4.35. A velocity field is specified in Lagrangian form by $v_1 = -X_2e^{-t}$, $v_2 = -X_3$, $v_3 = 2t$. Determine the acceleration components in Eulerian form. *Ans.* $a_1 = e^{-t}(x_2 + tx_3 - t^3)$, $a_2 = 0$, $a_3 = 2$

4.36. Show that the velocity field $v_i = \epsilon_{ijk}b_jx_k + c_i$ where b_i and c_i are constant vectors, represents a rigid body rotation and determine the vorticity vector for this motion.
Ans. $q_i = b_ix_{j,j} - b_i = 2b_i$

4.37. Show that for the flow $v_i = x_i/(1+t)$ the streamlines and path lines coincide.

4.38. The electrical field strength in a region containing a fluid flow is given by $\lambda = (A\cos 3t)/r$ where $r^2 = x_1^2 + x_2^2$ and A is a constant. The velocity field of the fluid is $v_1 = x_1^2x_2 + x_2^3$, $v_2 = -x_1^3 - x_1x_2^2$, $v_3 = 0$. Determine $d\lambda/dt$ at $P(x_1,x_2,x_3)$. *Ans.* $d\lambda/dt = (-3A\sin 3t)/r$

4.39. Show that for the velocity field $v_1 = x_1^2x_2 + x_2^3$, $v_2 = -x_1^3 - x_1x_2^2$, $v_3 = 0$ the streamlines are circular.

4.40. For the continuum motion $x_1 = X_1$, $x_2 = e^t(X_2+X_3)/2 + e^{-t}(X_2-X_3)/2$, $x_3 = e^t(X_2+X_3)/2 - e^{-t}(X_2-X_3)/2$, show that $D_{ij} = d\epsilon_{ij}/dt$ at $t=0$. Compare these tensors at $t = 0.5$.

4.41. For the velocity field $v_1 = x_1^2x_2 + x_2^3$, $v_2 = -(x_1^3 + x_1x_2^2)$, $v_3 = 0$, determine the principal axes and principal values of $\mathbf{D}$ at $P(1,2,3)$.

$$Ans.\quad D^*_{ij} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{pmatrix};\quad a_{ij} = \begin{pmatrix} 3/\sqrt{10} & 1/\sqrt{10} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{10} & -3/\sqrt{10} & 0 \end{pmatrix}$$

4.42. For the velocity field of Problem 4.41 determine the rate of extension in the direction $\hat{\nu} = (\hat{e}_1 - 2\hat{e}_2 + 2\hat{e}_3)/3$ at $P(1,2,3)$. What is the maximum shear rate at P?
Ans. $d^{(\hat{\nu})} = -24/9$, $\dot{\gamma}_{\max} = 5$

4.43. Show that $d(\partial x_i/\partial X_j)/dt = v_{i,k}x_{k,j}$ and use this to derive (*4.41*) of the text directly from (*4.38*).

4.44. Prove the identity $\epsilon_{pqr}(v_sv_{r,s})_{,q} = q_pv_{q,q} + v_qq_{p,q} - q_qv_{p,q}$ where v_i is the velocity and q_i the vorticity. Also show that $v_{i,j}v_{j,i} = D_{ij}D_{ij} - q_iq_i/2$.

4.45. Prove that the material derivative of the total vorticity is given by

$$\frac{d}{dt}\int_V q_i\,dV = \int_S [\epsilon_{ijk}a_k + q_jv_i]\,dS_j$$

Chapter 5

Fundamental Laws of Continuum Mechanics

5.1 CONSERVATION OF MASS. CONTINUITY EQUATION

Associated with every material continuum there is the property known as *mass*. The amount of mass in that portion of the continuum occupying the spatial volume V at time t is given by the integral

$$m = \int_V \rho(\mathbf{x}, t)\, dV \tag{5.1}$$

in which $\rho(\mathbf{x}, t)$ is a continuous function of the coordinates called the *mass density*. The law of *conservation of mass* requires that the mass of a specific portion of the continuum remain constant, and hence that the material derivative of (*5.1*) be zero. Therefore from (*4.52*) with $P^*_{ij\ldots}(\mathbf{x}, t) \equiv \rho(\mathbf{x}, t)$, the rate of change of m in (*5.1*) is

$$\frac{dm}{dt} = \frac{d}{dt}\int_V \rho(\mathbf{x}, t)\, dV = \int_V \left[\frac{d\rho}{dt} + \rho\frac{\partial v_k}{\partial x_k}\right] dV = 0 \tag{5.2}$$

Since this equation holds for an arbitrary volume V, the integrand must vanish, or

$$\frac{d\rho}{dt} + \rho v_{k,k} = 0 \quad \text{or} \quad \frac{d\rho}{dt} + \rho(\nabla \cdot \mathbf{v}) = 0 \tag{5.3}$$

This equation is called the *continuity equation*; using the material derivative operator it may be put into the alternative form

$$\frac{\partial\rho}{\partial t} + (\rho v_k)_{,k} = 0 \quad \text{or} \quad \frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0 \tag{5.4}$$

For an *incompressible* continuum the mass density of each particle is independent of time, so that $d\rho/dt = 0$ and (*5.3*) yields the result

$$v_{k,k} = 0 \quad \text{or} \quad \operatorname{div}\mathbf{v} = 0 \tag{5.5}$$

The velocity field $\mathbf{v}(\mathbf{x}, t)$ of an incompressible continuum can therefore be expressed by the equation

$$v_i = \epsilon_{ijk}s_{k,j} \quad \text{or} \quad \mathbf{v} = \nabla \times \mathbf{s} \tag{5.6}$$

in which $\mathbf{s}(\mathbf{x}, t)$ is called the *vector potential* of $\mathbf{v}$.

The continuity equation may also be expressed in the Lagrangian, or material form. The conservation of mass requires that

$$\int_{V_0} \rho_0(\mathbf{X}, 0)\, dV_0 = \int_V \rho(\mathbf{x}, t)\, dV \tag{5.7}$$

where the integrals are taken over the same particles, i.e. V is the volume now occupied by the material which occupied V_0 at time $t = 0$. Using (*4.1*) and (*4.38*), the right hand integral in (*5.7*) may be converted so that

$$\int_{V_0} \rho_0(\mathbf{X}, 0)\, dV_0 = \int_{V_0} \rho(\mathbf{x}(\mathbf{X}, t), t) J\, dV_0 = \int_{V_0} \rho(\mathbf{X}, t) J\, dV_0 \tag{5.8}$$

Since this relationship must hold for any volume V_0, it follows that

$$\rho_0 = \rho J \tag{5.9}$$

which implies that the product ρJ is independent of time since V is arbitrary, or that

$$\frac{d}{dt}(\rho J) = 0 \tag{5.10}$$

Equation (*5.10*) is the *Lagrangian differential form* of the continuity equation.

5.2 LINEAR MOMENTUM PRINCIPLE. EQUATIONS OF MOTION. EQUILIBRIUM EQUATIONS

A moving continuum which occupies the volume V at time t is shown in Fig. 5-1. Body forces b_i per unit mass are given. On the differential element dS of the bounding surface, the stress vector is $t_i^{(\hat{n})}$. The velocity field $v_i = du_i/dt$ is prescribed throughout the region occupied by the continuum. For this situation, the total *linear momentum* of the mass system within V is given by

$$P_i(t) = \int_V \rho v_i \, dV \tag{5.11}$$

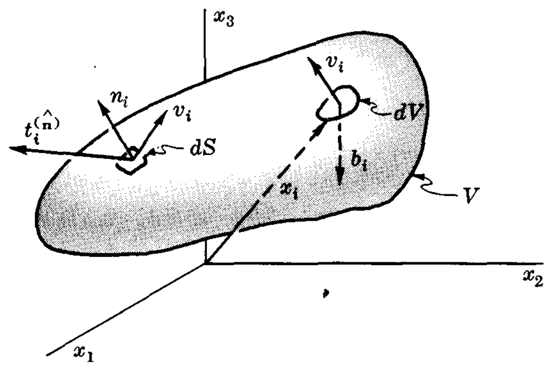


Fig. 5-1

Based upon Newton's second law, the *principle of linear momentum* states that the time rate of change of an arbitrary portion of a continuum is equal to the resultant force acting upon the considered portion. Therefore if the internal forces between particles of the continuum in Fig. 5-1 obey Newton's third law of action and reaction, the momentum principle for this mass system is expressed by

$$\int_S t_i^{(\hat{n})} \, dS + \int_V \rho b_i \, dV = \frac{d}{dt}\int_V \rho v_i \, dV$$

or

$$\int_S \mathbf{t}^{(\hat{n})} \, dS + \int_V \rho \mathbf{b} \, dV = \frac{d}{dt}\int_V \rho \mathbf{v} \, dV \tag{5.12}$$

Upon substituting $t_i^{(\hat{n})} = \sigma_{ji} n_j$ into the first integral of (*5.12*) and converting the resulting surface integral by the divergence theorem of Gauss, (*5.12*) becomes

$$\int_V (\sigma_{ji,j} + \rho b_i) \, dV = \frac{d}{dt}\int_V \rho v_i \, dV \quad \text{or} \quad \int_V (\nabla_{\mathbf{x}} \cdot \mathbf{\Sigma} + \rho \mathbf{b}) \, dV = \frac{d}{dt}\int \rho \mathbf{v} \, dV \tag{5.13}$$

In calculating the material derivative in (*5.13*), the continuity equation in the form given by (*5.10*) may be used. Thus

$$\frac{d}{dt}\int_V \rho v_i \, dV = \frac{d}{dt}\int_{V_0} \rho v_i J \, dV_0 = \int_{V_0}\left[v_i \frac{d(\rho J)}{dt} + \rho J \frac{dv_i}{dt}\right] dV_0 = \int_V \frac{dv_i}{dt}\rho \, dV \tag{5.14}$$

Replacing the right hand side of (*5.13*) by the right hand side of (*5.14*) and collecting terms results in the *linear momentum principle* in *integral form,*

$$\int_V (\sigma_{ji,j} + \rho b_i - \rho \dot{v}_i) \, dV = 0 \quad \text{or} \quad \int_V (\nabla_{\mathbf{x}} \cdot \mathbf{\Sigma} + \rho \mathbf{b} - \rho \dot{\mathbf{v}}) \, dV = 0 \tag{5.15}$$

Since the volume V is arbitrary, the integrand of (*5.15*) must vanish. The resulting equations,

$$\sigma_{ji,j} + \rho b_i = \rho \dot{v}_i \qquad \text{or} \qquad \nabla_{\mathbf{x}} \cdot \mathbf{\Sigma} + \rho \mathbf{b} = \rho \dot{\mathbf{v}} \tag{5.16}$$

are known as the *equations of motion.*

The important case of static equilibrium, in which the acceleration components vanish, is given at once from (*5.16*) as

$$\sigma_{ji,j} + \rho b_i = 0 \qquad \text{or} \qquad \nabla_{\mathbf{x}} \cdot \mathbf{\Sigma} + \rho \mathbf{b} = 0 \tag{5.17}$$

These are the *equilibrium equations,* used extensively in solid mechanics.

5.3 MOMENT OF MOMENTUM (ANGULAR MOMENTUM) PRINCIPLE

The *moment of momentum* is, as the name implies, simply the *moment of linear momentum* with respect to some point. Thus for the continuum shown in Fig. 5-1, the total moment of momentum or *angular momentum* as it is often called, with respect to the origin, is

$$N_i(t) = \int_V \epsilon_{ijk} x_j \rho v_k \, dV \qquad \text{or} \qquad \mathbf{N} = \int_V (\mathbf{x} \times \rho \mathbf{v}) \, dV \tag{5.18}$$

in which x_j is the position vector of the volume element dV. The *moment of momentum principle* states that the time rate of change of the angular momentum of any portion of a continuum with respect to an arbitrary point is equal to the resultant moment (with respect to that point) of the body and surface forces acting on the considered portion of the continuum. Accordingly, for the continuum of Fig. 5-1, the moment of momentum principle is expressed in integral form by

$$\int_S \epsilon_{ijk} x_j t_k^{(\hat{\mathbf{n}})} \, dS + \int_V \epsilon_{ijk} x_j \rho b_k \, dV = \frac{d}{dt} \int_V \epsilon_{ijk} x_j \rho v_k \, dV$$

or

$$\int_S (\mathbf{x} \times \mathbf{t}^{(\hat{\mathbf{n}})}) \, dS + \int_V (\mathbf{x} \times \rho \mathbf{b}) \, dV = \frac{d}{dt} \int_V (\mathbf{x} \times \rho \mathbf{v}) \, dV \tag{5.19}$$

Equation (*5.19*) is valid for those continua in which the forces between particles are equal, opposite and collinear, and in which distributed moments are absent.

The moment of momentum principle does not furnish any new differential equation of motion. If the substitution $t_k^{(\hat{\mathbf{n}})} = \sigma_{pk} n_p$ is made in (*5.19*), and the symmetry of the stress tensor assumed, the equation is satisfied identically by using the relationship given in (*5.16*). If stress symmetry is not assumed, such symmetry may be shown to follow directly from (*5.19*), which upon substitution of $t_k^{(\hat{\mathbf{n}})} = \sigma_{pk} n_p$, reduces to

$$\int_V \epsilon_{ijk} \sigma_{jk} \, dV = 0 \qquad \text{or} \qquad \int_V \mathbf{\Sigma}_v \, dV = 0 \tag{5.20}$$

Since the volume V is arbitrary,

$$\epsilon_{ijk} \sigma_{jk} = 0 \qquad \text{or} \qquad \mathbf{\Sigma}_v = 0 \tag{5.21}$$

which by expansion demonstrates that $\sigma_{jk} = \sigma_{kj}$.

5.4 CONSERVATION OF ENERGY. FIRST LAW OF THERMODYNAMICS. ENERGY EQUATION

If mechanical quantities only are considered, the *principle of conservation of energy* for the continuum of Fig. 5-1 may be derived directly from the equation of motion given by (*5.16*). To accomplish this, the scalar product between (*5.16*) and the velocity v_i is first computed, and the result integrated over the volume V. Thus

$$\int_V \rho v_i \dot{v}_i \, dV = \int_V v_i \sigma_{ji,j} \, dV + \int_V \rho v_i b_i \, dV \tag{5.22}$$

But
$$\int_V \rho v_i \dot{v}_i \, dV = \frac{d}{dt} \int_V \rho \frac{v_i v_i}{2} \, dV = \frac{d}{dt} \int_V \frac{\rho v^2}{2} \, dV = \frac{dK}{dt} \tag{5.23}$$

which represents the time rate of change of the *kinetic energy* K in the continuum. Also, $v_i \sigma_{ji,j} = (v_i \sigma_{ji})_{,j} - v_{i,j} \sigma_{ji}$ and by (*4.19*) $v_{i,j} = D_{ij} + V_{ij}$, so that (*5.22*) may be written

$$\frac{dK}{dt} + \int_V D_{ij} \sigma_{ji} \, dV = \int_V (v_i \sigma_{ji})_{,j} \, dV + \int_V \rho v_i b_i \, dV \tag{5.24}$$

since $V_{ij} \sigma_{ji} = 0$. Finally, converting the first integral on the right hand side of (*5.24*) to a surface integral by the divergence theorem of Gauss, and making use of the identity $t_i^{(\hat{n})} = \sigma_{ji} n_j$, the *energy equation* for a continuum appears in the form

$$\frac{dK}{dt} + \int_V D_{ij} \sigma_{ji} \, dV = \int_S v_i t_i^{(\hat{n})} \, dS + \int_V \rho b_i v_i \, dV \tag{5.25}$$

This equation relates the time rate of change of total mechanical energy of the continuum on the left side to the rate of work done by the surface and body forces on the right hand side of the equation. The integral on the left side is known as the time rate of change of *internal mechanical energy*, and written dU/dt. Therefore (*5.25*) may be written briefly as

$$\frac{dK}{dt} + \frac{dU}{dt} = \frac{đW}{dt} \tag{5.26}$$

where $đW/dt$ represents the rate of work, and the special symbol $đ$ is used to indicate that this quantity is *not* an *exact* differential.

If both mechanical and non-mechanical energies are to be considered, the principle of conservation of energy in its most general form must be used. In this form the conservation principle states that *the time rate of change of the kinetic plus the internal energy is equal to the sum of the rate of work plus all other energies supplied to, or removed from the continuum per unit time.* Such energies supplied may include *thermal* energy, *chemical* energy, or *electromagnetic* energy. In the following, only *mechanical* and *thermal* energies are considered, and the energy principle takes on the form of the well-known *first law of thermodynamics*.

For a thermomechanical continuum it is customary to express the time rate of change of internal energy by the integral expression

$$\frac{dU}{dt} = \frac{d}{dt} \int_V \rho u \, dV = \int_V \rho \dot{u} \, dV \tag{5.27}$$

where u is called the *specific internal energy*. (The symbol u for specific energy is so well established in the literature that it is used in the energy equations of this chapter since there appears to be only a negligible chance that it will be mistaken in what follows for the magnitude of the displacement vector u_i.) Also, if the vector c_i is defined as the *heat flux* per unit area per unit time by conduction, and z is taken as the *radiant heat constant* per unit mass per unit time, the rate of increase of total heat into the continuum is given by

$$\frac{đQ}{dt} = -\int_S c_i n_i \, dS + \int_V \rho z \, dV \tag{5.28}$$

Therefore the energy principle for a thermomechanical continuum is given by

$$\frac{dK}{dt} + \frac{dU}{dt} = \frac{đW}{dt} + \frac{đQ}{dt} \tag{5.29}$$

or, in terms of the energy integrals, as

$$\frac{d}{dt}\int_V \rho\frac{v_iv_i}{2}\,dV + \int_V \rho\dot{u}\,dV = \int_S t_i^{(\hat{n})}v_i\,dS + \int_V \rho v_i b_i\,dV + \int_V \rho z\,dV - \int_S c_i n_i\,dS \tag{5.30}$$

Converting the surface integrals in (*5.30*) to volume integrals by the divergence theorem of Gauss, and again using the fact that V is arbitrary, leads to the local form of the energy equation:

$$\frac{d}{dt}\left(\frac{v^2}{2}+u\right) = \frac{1}{\rho}(\sigma_{ij}v_i)_{,j} + b_iv_i - \frac{1}{\rho}c_{i,i} + z$$

or

$$\frac{du}{dt} = \frac{1}{\rho}\boldsymbol{\Sigma}:\mathbf{D} - \frac{1}{\rho}\nabla\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{v} + z \tag{5.31}$$

Within the arbitrarily small volume element for which the local energy equation (*5.31*) is valid, the balance of momentum given by (*5.16*) must also hold. Therefore by taking the scalar product between (*5.16*) and the velocity $\rho\dot{v}_iv_i = v_i\sigma_{ji,j} + \rho v_ib_i$ and, after some simple manipulations, subtracting this product from (*5.31*), the result is the reduced, but highly useful form of the local energy equation,

$$\frac{du}{dt} = \frac{1}{\rho}\sigma_{ij}D_{ij} - \frac{1}{\rho}c_{i,i} + z \tag{5.32}$$

This equation expresses the rate of change of *internal energy* as the sum of the *stress power* plus the *heat* added to the continuum.

5.5 EQUATIONS OF STATE. ENTROPY. SECOND LAW OF THERMODYNAMICS

The complete characterization of a thermodynamic system (here, a continuum) is said to describe the *state* of the system. This description is specified, in general, by several thermodynamic and kinematic quantities called *state variables.* A change with time of the state variables characterizes a *thermodynamic process.* The state variables used to describe a given system are usually not all independent. Functional relationships exist among the state variables and these relationships are expressed by the so-called *equations of state.* Any state variable which may be expressed as a single-valued function of a set of other state variables is known as a *state function.*

As presented in the previous section, the first law of thermodynamics postulates the interconvertibility of mechanical and thermal energy. The relationship expressing conversion of heat and work into kinetic and internal energies during a thermodynamic process is set forth in the energy equation. The first law, however, leaves unanswered the question of the extent to which the conversion process is *reversible* or *irreversible.* All real processes are irreversible, but the reversible process is a very useful hypothesis since energy dissipation may be assumed negligible in many situations. The basic criterion for irreversibility is given by the *second law of thermodynamics* through its statement on the limitations of *entropy production.*

The second law of thermodynamics postulates the existence of two distinct state functions; T the *absolute temperature,* and S the *entropy,* with certain following properties. T is a positive quantity which is a function of the empirical temperature θ, only. The entropy is an extensive property, i.e. the total entropy in the system is the sum of the entropies of its parts. In continuum mechanics the *specific entropy* (per unit mass), or *entropy density* is denoted by s, so that the total entropy L is given by $L = \int_V \rho s\,dV$. The entropy of a system can change either by interactions that occur with the surroundings, or by changes that take place within the system. Thus

$$ds = ds^{(e)} + ds^{(i)} \tag{5.33}$$

where ds is the increase in specific entropy, $ds^{(e)}$ is the increase due to interaction with the exterior, and $ds^{(i)}$ is the internal increase. The change $ds^{(i)}$ is never negative. It is zero for a reversible process, and positive for an irreversible process. Therefore

$$ds^{(i)} > 0 \quad \text{(irreversible process)} \tag{5.34}$$

$$ds^{(i)} = 0 \quad \text{(reversible process)} \tag{5.35}$$

In a reversible process, if $dq_{(R)}$ denotes the heat supplied per unit mass to the system, the change $ds^{(e)}$ is given by

$$ds^{(e)} = \frac{dq_{(R)}}{T} \quad \text{(reversible process)} \tag{5.36}$$

5.6 THE CLAUSIUS-DUHEM INEQUALITY. DISSIPATION FUNCTION

According to the second law, the time rate of change of total entropy L in a continuum occupying a volume V is never less than the sum of the entropy influx through the continuum surface plus the entropy produced internally by body sources. Mathematically, this entropy principle may be expressed in integral form as the *Clausius-Duhem inequality*,

$$\frac{d}{dt}\int_V \rho s\, dV \geqq \int_V \rho e\, dV - \int_S \frac{c_i n_i}{T}\, dS \tag{5.37}$$

where e is the local entropy source per unit mass. The equality in (*5.37*) holds for reversible processes; the inequality applies to irreversible processes.

The Clausius-Duhem inequality is valid for arbitrary choice of volume V so that transforming the surface integral in (*5.37*) by the divergence theorem of Gauss, the local form of the *internal entropy production rate* γ, per unit mass, is given by

$$\gamma \equiv \frac{ds}{dt} - e - \frac{1}{\rho}\left(\frac{c_i}{T}\right)_{,i} \geqq 0 \tag{5.38}$$

This inequality must be satisfied for every process and for any assignment of state variables. For this reason it plays an important role in imposing restrictions upon the so-called constitutive equations discussed in the following section.

In much of continuum mechanics, it is often assumed (based upon statistical mechanics of irreversible processes) that the stress tensor may be split into two parts according to the scheme,

$$\sigma_{ij} = \sigma_{ij}^{(C)} + \sigma_{ij}^{(D)} \tag{5.39}$$

where $\sigma_{ij}^{(C)}$ is a *conservative* stress tensor, and $\sigma_{ij}^{(D)}$ is a *dissipative* stress tensor. With this assumption the energy equation (*5.32*) may be written with the use of (*4.25*) as

$$\frac{du}{dt} = \frac{1}{\rho}\sigma_{ij}^{(C)}\dot{\epsilon}_{ij} + \frac{1}{\rho}\sigma_{ij}^{(D)}\dot{\epsilon}_{ij} + \frac{dq}{dt} \tag{5.40}$$

In this equation, $\frac{1}{\rho}\sigma_{ij}^{(D)}\dot{\epsilon}_{ij}$ is the rate of energy dissipated per unit mass by the stress, and dq/dt is the rate of heat influx per unit mass into the continuum. If the continuum undergoes a reversible process, there will be no energy dissipation, and furthermore, $dq/dt = dq_{(R)}/dt$, so that (*5.40*) and (*5.36*) may be combined to yield

$$\frac{du}{dt} = \frac{1}{\rho}\sigma_{ij}^{(C)}\dot{\epsilon}_{ij} + T\frac{ds}{dt} \tag{5.41}$$

Therefore in the irreversible process described by (*5.40*), the entropy production rate may be expressed by inserting (*5.41*). Thus

$$\frac{ds}{dt} = \frac{1}{T}\frac{dq}{dt} + \frac{1}{\rho T}\sigma_{ij}^{(D)}\dot{\epsilon}_{ij} \tag{5.42}$$

The scalar $\sigma_{ij}^{(D)}\dot{\epsilon}_{ij}$ is called the *dissipation function.* For an irreversible, adiabatic process ($dq = 0$), $ds/dt > 0$ by the second law, so from (*5.42*) it follows that the dissipation function is *positive definite,* since both ρ and T are always positive.

5.7 CONSTITUTIVE EQUATIONS. THERMOMECHANICAL AND MECHANICAL CONTINUA

In the preceding sections of this chapter, several equations have been developed that must hold for every process or motion that a continuum may undergo. For a thermomechanical continuum in which the mechanical and thermal phenomena are coupled, the basic equations are

(*a*) the *equation* of *continuity,* (*5.4*)

$$\frac{\partial\rho}{\partial t} + (\rho v_k)_{,k} = 0 \qquad \text{or} \qquad \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{5.43}$$

(*b*) the *equation* of *motion,* (*5.16*)

$$\sigma_{ji,j} + \rho b_i = \rho\dot{v}_i \qquad \text{or} \qquad \nabla_{\mathbf{x}}\cdot\mathbf{\Sigma} + \rho\mathbf{b} = \rho\dot{\mathbf{v}} \tag{5.44}$$

(*c*) the *energy equation,* (*5.32*)

$$\frac{du}{dt} = \frac{1}{\rho}\sigma_{ij}D_{ij} - \frac{1}{\rho}c_{i,i} + z \qquad \text{or} \qquad \frac{du}{dt} = \frac{1}{\rho}\mathbf{\Sigma}:\mathbf{D} - \rho\nabla\cdot\mathbf{c} + z \tag{5.45}$$

Assuming that body forces b_i and the distributed heat sources z are prescribed, (*5.43*), (*5.44*) and (*5.45*) consist of *five* independent equations involving *fourteen* unknown functions of *time* and *position.* The unknowns are the *density* ρ, the three *velocity* components v_i, (or, alternatively, the *displacement* components u_i), the six independent *stress* components σ_{ij}, the three components of the *heat flux* vector c_i, and the specific internal energy u. In addition, the Clausius-Duhem inequality (*5.38*)

$$\frac{ds}{dt} - e - \frac{1}{\rho}\left(\frac{q_i}{T}\right)_{,i} \geqq 0 \tag{5.46}$$

which governs entropy production, must hold. This introduces two additional unknowns: the *entropy density* s, and T, the *absolute temperature.* Therefore *eleven* additional equations must be supplied to make the system determinate. Of these, *six* will be in the form known as *constitutive equations,* which characterize the particular physical properties of the continuum under study. Of the remaining five, *three* will be in the form of *temperature-heat conduction* relations, and *two* will appear as *thermodynamic equations of state*; for example, perhaps as the *caloric* equation of state and the *entropic* equation of state. Specific formulation of the thermomechanical continuum problem is given in a subsequent chapter.

It should be pointed out that the function of the constitutive equations is to establish a mathematical relationship among the statical, kinematical and thermal variables, which will describe the behavior of the material when subjected to applied mechanical or thermal forces. Since real materials respond in an extremely complicated fashion under various loadings, constitutive equations do not attempt to encompass all the observed phenomena related to a particular material, but, rather, to define certain *ideal materials,* such as the ideal elastic solid or the ideal viscous fluid. Such idealizations or *material models* as they are sometimes called, are very useful in that they portray reasonably well over a definite range of loads and temperatures the behavior of real substances.

In many situations the interaction of mechanical and thermal processes may be neglected. The resulting analysis is known as the *uncoupled thermoelastic theory* of continua. Under this assumption the purely mechanical processes are governed by (*5.43*) and (*5.44*) since the energy equation (*5.45*) for this case is essentially a first integral of the equation of motion. The system of equations formed by (*5.43*) and (*5.44*) consists of *four* equations involving *ten* unknowns. *Six* constitutive equations are required to make the system determinate. In the uncoupled theory, the constitutive equations contain only the *statical* (stresses) and *kinematic* (velocities, displacements, strains) variables and are often referred to as *stress-strain relations*. Also, in the uncoupled theory, the temperature field is usually regarded as known, or at most, the heat-conduction problem must be solved separately and independently from the mechanical problem. In *isothermal* problems the temperature is assumed uniform and the problem is purely mechanical.

Solved Problems

CONTINUITY EQUATION (Sec. 5.1)

5.1. An *irrotational* motion of a continuum is described in Chapter 4 as one for which the vorticity vanishes identically. Determine the form of the continuity equation for such motions.

By (*4.29*), curl $\mathbf{v} = 0$ when $\mathbf{q} \equiv 0$, and so $\mathbf{v}$ becomes the gradient of a scalar field $\phi(x_i, t)$ (see Problem 1.50). Thus $v_i = \phi_{,i}$ and (*5.3*) is now $d\rho/dt + \rho\phi_{,kk} = 0$ or $d\rho/dt + \rho\nabla^2\phi = 0$.

5.2. If $P^{**}_{ij\ldots}(\mathbf{x}, t)$ represents any scalar, vector or tensor property per unit mass of a continuum so that $P^{*}_{ij\ldots}(\mathbf{x}, t) = \rho P^{**}_{ij\ldots}(\mathbf{x}, t)$ show that

$$\frac{d}{dt}\int_V \rho P^{**}_{ij\ldots}(\mathbf{x}, t)\,dV = \int_V \rho\,\frac{dP^{**}_{ij\ldots}(\mathbf{x}, t)}{dt}\,dV$$

By (*4.52*),

$$\frac{d}{dt}\int_V \rho P^{**}_{ij\ldots}\,dV = \int_V \left[\frac{d}{dt}(\rho P^{**}_{ij\ldots}) + \rho P^{**}_{ij\ldots}\frac{\partial v_k}{\partial x_k}\right] dV$$

$$= \int_V \left[\rho\frac{dP^{**}_{ij\ldots}}{dt} + P^{**}_{ij\ldots}\left(\frac{d\rho}{dt} + \rho\frac{\partial v_k}{\partial x_k}\right)\right] dV = \int_V \rho\frac{dP^{**}_{ij\ldots}}{dt}\,dV$$

since by (*5.3*), $d\rho/dt + \rho v_{k,k} = 0$.

5.3. Show that the material form $d(\rho J)/dt = 0$ of the continuity equation and the spatial form $d\rho/dt + \rho v_{k,k} = 0$ are equivalent.

Differentiating, $d(\rho J)/dt = (d\rho/dt)J + \rho\,dJ/dt = 0$ and from Problem 4.28, $dJ/dt = Jv_{k,k}$ so that $d(\rho J)/dt = J(d\rho/dt + \rho v_{k,k}) = 0$.

5.4. Show that the velocity field $v_i = Ax_i/r^3$, where $x_i x_i = r^2$ and A is an arbitrary constant, satisfies the continuity equation for an incompressible flow.

From (*5.5*) $v_{k,k} = 0$ for incompressible flow. Here

$$v_{i,k} = A(x_{i,k}/r^3 - 3x_i x_k/r^5) = A(\delta_{ik}/r^3 - 3x_i x_k/r^5)$$

and so $v_{k,k} = (3-3)/r^3 = 0$ to satisfy the continuity equation.

5.5. For the velocity field $v_i = x_i/(1+t)$, show that $\rho x_1 x_2 x_3 = \rho_0 X_1 X_2 X_3$.

Here $v_{k,k} = 3/(1+t)$ and integrating (*5.3*) yields $\ln \rho = -\ln(1+t)^3 + \ln C$ where C is a constant of integration. Since $\rho = \rho_0$ when $t = 0$, this equation becomes $\rho = \rho_0/(1+t)^3$. Next by integrating the velocity field $dx_i/x_i = dt/(1+t)$ (no sum on i), $x_i = X_i/(1+t)$ and hence $\rho x_1 x_2 x_3 = \rho_0 X_1 X_2 X_3$.

LINEAR AND ANGULAR MOMENTUM. EQUATIONS OF MOTION (Sec. 5.2-5.3)

5.6. Show by a direct expansion of each side that the identity $\epsilon_{ijk}\sigma_{jk}\hat{\mathbf{e}}_i = \mathbf{\Sigma}_v$ used in (*5.20*) and (*5.21*) is valid.

By (*1.15*) and (*2.8*),

$$\begin{aligned}\mathbf{\Sigma}_v &= \sigma_{11}\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_1 + \sigma_{12}\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 + \sigma_{13}\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_3 + \cdots + \sigma_{33}\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_3 \\ &= (\sigma_{23} - \sigma_{32})\hat{\mathbf{e}}_1 + (\sigma_{31} - \sigma_{13})\hat{\mathbf{e}}_2 + (\sigma_{12} - \sigma_{21})\hat{\mathbf{e}}_3\end{aligned}$$

Also, expanding $\epsilon_{ijk}\sigma_{jk}$ gives identical results, $(\sigma_{23} - \sigma_{32})$ for $i = 1$, $(\sigma_{31} - \sigma_{13})$ for $i = 2$, $(\sigma_{12} - \sigma_{21})$ for $i = 3$.

5.7. If distributed body moments m_i per unit volume act throughout a continuum, show that the equations of motion (*5.16*) remain valid but the stress tensor can no longer be assumed symmetric.

Since (*5.16*) is derived on the basis of *force* equilibrium, it is not affected. Now, however, (*5.19*) acquires an additional term so that

$$\frac{d}{dt}\int_V \epsilon_{ijk} x_j \rho v_k \, dV = \int_S \epsilon_{ijk} x_j t_k^{(\hat{n})} \, dS + \int_V (\epsilon_{ijk} x_j \rho b_k + m_i) \, dV$$

which reduces to (see Problem 2.9) $\int_V (\epsilon_{ijk}\sigma_{jk} + m_i)\, dV = 0$, and because V is arbitrary, $\epsilon_{ijk}\sigma_{jk} + m_i = 0$ for this case.

5.8. The momentum principle in differential form (the so-called local or "in the small" form) is expressed by the equation $\partial(\rho v_i)/\partial t = \rho b_i + (\sigma_{ij} - \rho v_i v_j)_{,j}$. Show that the equation of motion (*5.16*) follows from this equation.

Carrying out the indicated differentiation and rearranging the terms in the resulting equation yields

$$v_i(\partial \rho/\partial t + \rho_{,j} v_j + \rho v_{j,j}) + \rho(\partial v_i/\partial t + v_j v_{i,j}) = \rho b_i + \sigma_{ij,j}$$

The first term on the left is zero by (*5.4*) and the second term is ρa_i. Thus $\rho a_i = \rho b_i + \sigma_{ij,j}$ which is (*5.16*).

5.9. Show that (*5.19*) reduces to (*5.20*).

Substituting $\sigma_{pk} n_p$ for $t_k^{(\hat{n})}$ in (*5.19*) and applying the divergence theorem (*1.157*) to the resulting surface integral gives

$$\int_V \epsilon_{ijk}\{(x_j \sigma_{pk})_{,p} + x_j \rho b_k\} \, dV = \frac{d}{dt}\int_V \epsilon_{ijk} \rho (x_j v_k) \, dV$$

Using the results of Problem 5.2, the indicated differentiations here lead to

$$\int_V \epsilon_{ijk}\{x_{j,p}\sigma_{pk} + x_j(\sigma_{pk,p} + \rho b_k - \rho \dot{v}_k) - \rho v_j v_k\} \, dV = 0$$

The term in parentheses is zero by (*5.16*), also $x_{j,p} = \delta_{jp}$ and $\epsilon_{ijk} v_j v_k = 0$, so that finally $\int_V \epsilon_{ijk}\sigma_{jk} \, dV = 0$.

5.10. For a rigid body rotation about a point, $v_i = \epsilon_{ijk}\omega_j x_k$. Show that for this velocity (*5.19*) reduces to the well-known momentum principle of rigid body dynamics.

The left hand side of (*5.19*) is the total moment M_i of all surface and body forces relative to the origin. Thus for $v_i = \epsilon_{ijk}\omega_j x_k$,

$$M_i = \frac{d}{dt}\int_V \epsilon_{ijk}x_j\rho\epsilon_{kpq}\omega_p x_q\,dV = \frac{d}{dt}\int_V \omega_p\rho(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})x_j x_q\,dV$$

$$= \frac{d}{dt}\left[\omega_p\int_V \rho(\delta_{ip}x_q x_q - x_p x_i)\,dV\right] = \frac{d}{dt}(\omega_p I_{ip})$$

where $I_{ip} = \int_V \rho(\delta_{ip}x_q x_q - x_p x_i)\,dV$ is the moment of inertia tensor.

ENERGY. ENTROPY. DISSIPATION FUNCTION (Sec. 5.4-5.6)

5.11. Show that for a rigid body rotation with $v_i = \epsilon_{ijk}\omega_j x_k$, the kinetic energy integral of (*5.23*) reduces to the familiar form given in rigid body dynamics.

From (*5.23*),

$$K = \int_V \rho\frac{v_i v_i}{2}\,dV = \frac{1}{2}\int_V \rho\epsilon_{ijk}\omega_j x_k\epsilon_{ipq}\omega_p x_q\,dV$$

$$= \frac{1}{2}\int_V \rho\omega_p\omega_j(\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp})x_k x_q\,dV$$

$$= \frac{\omega_j\omega_p}{2}\int_V \rho(\delta_{jp}x_q x_q - x_p x_j)\,dV = \frac{\omega_j\omega_p I_{jp}}{2}$$

In symbolic notation note that $K = \dfrac{\boldsymbol{\omega}\cdot\mathbf{I}\cdot\boldsymbol{\omega}}{2}$.

5.12. At a certain point in a continuum the rate of deformation and stress tensors are given by

$$D_{ij} = \begin{pmatrix} 1 & 6 & 4 \\ 6 & 3 & 2 \\ 4 & 2 & 5 \end{pmatrix} \quad \text{and} \quad \sigma_{ij} = \begin{pmatrix} 4 & 0 & -1 \\ 0 & -2 & 7 \\ -1 & 7 & 8 \end{pmatrix}$$

Determine the value λ of the stress power $D_{ij}\sigma_{ij}$ at the point.

Multiplying each element of D_{ij} by its counterpart in σ_{ij} and adding, $\lambda = 4 + 0 - 4 + 0 - 6 + 14 - 4 + 14 + 40 = 58$.

5.13. If $\sigma_{ij} = -p\delta_{ij}$ where p is a positive constant, show that the stress power may be expressed by the equation $D_{ij}\sigma_{ij} = \dfrac{p}{\rho}\dfrac{d\rho}{dt}$.

By (*4.19*), $D_{ij} = v_{i,j} - V_{ij}$; and since $V_{ij}\sigma_{ij} = 0$, it follows that $D_{ij}\sigma_{ij} = v_{i,j}(-p\delta_{ij}) = -pv_{i,i}$. From the continuity equation (*5.3*), $v_{i,i} = -(1/\rho)(d\rho/dt)$ and so $D_{ij}\sigma_{ij} = (p/\rho)(d\rho/dt)$ for $\sigma_{ij} = -p\delta_{ij}$.

5.14. Determine the form of the energy equation if $\sigma_{ij} = (-p + \lambda^* D_{kk})\delta_{ij} + 2\mu^* D_{ij}$ and the heat conduction obeys the Fourier law $c_i = -kT_{,i}$.

From (*5.32*),

$$\rho\frac{du}{dt} = (-p + \lambda^* D_{kk})\delta_{ij}D_{ij} + 2\mu^* D_{ij}D_{ij} + kT_{,ii} + z$$

$$= \frac{p}{\rho}\frac{d\rho}{dt} + (\lambda^* + 2\mu^*)(\mathrm{I_D})^2 - 4\mu^*\mathrm{II_D} + kT_{,ii} + z$$

where $\mathrm{I_D}$ and $\mathrm{II_D}$ are the first and second invariants respectively of the rate of deformation tensor.

5.15. If $\sigma_{ij} = -p\delta_{ij}$, determine an equation for the rate of change of specific entropy during a reversible thermodynamic process.

Here $\sigma_{ij} = \sigma_{ij}^{(C)}$ and *(5.41)* gives $T\dfrac{ds}{dt} = \dfrac{du}{dt} + \dfrac{p}{\rho^2}\dfrac{d\rho}{dt}$ upon use of the result in Problem 5.13.

5.16. For the stress having $\sigma_{ij}^{(D)} = \beta D_{ik}D_{kj}$, determine the dissipation function in terms of the invariants of the rate of deformation tensor **D**.

Here by *(4.25)*, $\sigma_{ij}^{(D)}\dot{\epsilon}_{ij} = \beta D_{ik}D_{kj}D_{ij}$ which is the trace of $\mathbf{D}^3$ (see page 16) and may be evaluated by using the principal axis values $D_{(1)}, D_{(2)}, D_{(3)}$. Thus by *(1.138)* the trace

$$\begin{aligned} D_{ij}D_{ik}D_{kj} &= D_{(1)}^3 + D_{(2)}^3 + D_{(3)}^3 \\ &= (D_{(1)} + D_{(2)} + D_{(3)})^3 - 3(D_{(1)} + D_{(2)} + D_{(3)})(D_{(1)}D_{(2)} + D_{(2)}D_{(3)} + D_{(3)}D_{(1)}) \\ &\qquad + 3D_{(1)}D_{(2)}D_{(3)} \end{aligned}$$

Therefore $\sigma_{ij}^{(D)}\dot{\epsilon}_{ij} = \beta[\mathrm{I}_{\mathbf{D}}^3 - 3\mathrm{I}_{\mathbf{D}}\mathrm{II}_{\mathbf{D}} + 3\mathrm{III}_{\mathbf{D}}]$.

CONSTITUTIVE EQUATIONS (Sec. 5.7)

5.17. For the constitutive equations $\sigma_{ij} = K_{ijpq}D_{pq}$ show that because of the symmetry of the stress and rate of deformation tensors the fourth order tensor K_{ijpq} has at most 36 distinct components. Display the components in a 6×6 array.

Since $\sigma_{ij} = \sigma_{ji}$, $K_{ijpq} = K_{jipq}$; and since $D_{ij} = D_{ji}$, $K_{ijpq} = K_{ijqp}$. If K_{ijpq} is considered as the outer product of two symmetric tensors $A_{ij}B_{pq} = K_{ijpq}$, it is clear that since both A_{ij} and B_{ij} have six independent components, K_{ijpq} will have at most 36 distinct components.

The usual arrangement followed in displaying the components of K_{ijpq} is

$$K_{ijpq} = \begin{pmatrix} K_{1111} & K_{1122} & K_{1133} & K_{1123} & K_{1131} & K_{1112} \\ K_{2211} & K_{2222} & K_{2233} & K_{2223} & K_{2231} & K_{2212} \\ K_{3311} & K_{3322} & K_{3333} & K_{3323} & K_{3331} & K_{3312} \\ K_{2311} & K_{2322} & K_{2333} & K_{2323} & K_{2331} & K_{2312} \\ K_{3111} & K_{3122} & K_{3133} & K_{3123} & K_{3131} & K_{3112} \\ K_{1211} & K_{1222} & K_{1233} & K_{1223} & K_{1231} & K_{1212} \end{pmatrix}$$

5.18. If the continuum having the constitutive relations $\sigma_{ij} = K_{ijpq}D_{pq}$ of Problem 5.17 is assumed *isotropic* so that K_{ijpq} has the same array of components in any rectangular Cartesian system of axes, show that by a cyclic labeling of the coordinate axes the 36 components may be reduced to 26.

The coordinate directions may be labeled in six different ways as shown in Fig. 5-2. Isotropy of K_{ijpq} then requires that $K_{1122} = K_{1133} = K_{2233} = K_{2211} = K_{3311} = K_{3322}$ and that $K_{1212} = K_{1313} = K_{2323} = K_{2121} = K_{3131} = K_{3232}$ which reduces the 36 components to 26. By suitable reflections and rotations of the coordinate axes these 26 components may be reduced to 2 for the case of isotropy.

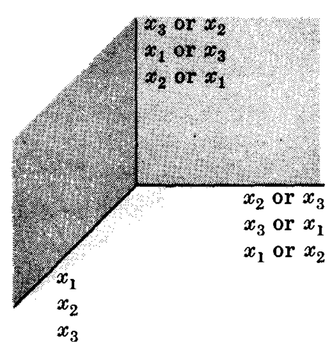


Fig. 5-2

5.19. For isotropy K_{ijpq} may be represented by $K_{ijpq} = \lambda^*\delta_{ij}\delta_{pq} + \mu^*(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp})$. Use this to develop the constitutive equation $\sigma_{ij} = K_{ijpq}D_{pq}$ in terms of λ^* and μ^*.

$$\begin{aligned} \sigma_{ij} &= \lambda^*\delta_{ij}\delta_{pq}D_{pq} + \mu^*(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp})D_{pq} \\ &= \lambda^*\delta_{ij}D_{pp} + \mu^*(D_{ij} + D_{ji}) = \lambda^*\delta_{ij}D_{pp} + 2\mu^*D_{ij} \end{aligned}$$

5.20. Show that the constitutive equation of Problem 5.19 may be split into the equivalent equations $\sigma_{ii} = (3\lambda^* + 2\mu^*)D_{ii}$ and $s_{ij} = 2\mu^* D'_{ij}$ where s_{ij} and D'_{ij} are the deviator tensors of stress and rate of deformation, respectively.

Substituting $\sigma_{ij} = s_{ij} + \delta_{ij}\sigma_{kk}/3$ and $D_{ij} = D'_{ij} + \delta_{ij}D_{kk}/3$ into $\sigma_{ij} = \lambda^*\delta_{ij}D_{kk} + 2\mu^* D_{ij}$ of Problem 5.19 results in the equation $s_{ij} + \delta_{ij}\sigma_{kk}/3 = \lambda^*\delta_{ij}D_{kk} + 2\mu^*(D'_{ij} + \delta_{ij}D_{kk}/3)$. From this when $i \neq j$, $s_{ij} = 2\mu^* D'_{ij}$ and hence $\sigma_{kk} = (3\lambda^* + 2\mu^*)D_{kk}$.

MISCELLANEOUS PROBLEMS

5.21. Show that $\dfrac{d}{dt}\left(\dfrac{q_i}{\rho}\right) = (\epsilon_{ijk}a_{k,j} + q_j v_{i,j})/\rho$ where ρ is the density, a_i the acceleration and q_i the vorticity vector.

By direct differentiation $\dfrac{d}{dt}\left(\dfrac{q_i}{\rho}\right) = \dfrac{\dot{q}_i}{\rho} - \dfrac{q_i\dot{\rho}}{\rho^2}$. But $\dot{q}_i = \epsilon_{ijk}a_{k,j} + q_j v_{i,j} - q_i v_{j,j}$ (see Problem 4.32); and by the continuity equation *(5.3)*, $\dot{\rho} = -\rho v_{i,i}$. Thus

$$\frac{d}{dt}\left(\frac{q_i}{\rho}\right) = \frac{1}{\rho}(\epsilon_{ijk}a_{k,j} + q_j v_{i,j} - q_i v_{j,j} + q_i v_{j,j}) = (\epsilon_{ijk}a_{k,j} + q_j v_{i,j})/\rho$$

5.22. A two dimensional incompressible flow is given by $v_1 = A(x_1^2 - x_2^2)/r^4$, $v_2 = A(2x_1x_2)/r^4$, $v_3 = 0$, where $r^2 = x_1^2 + x_2^2$. Show that the continuity equation is satisfied by this motion.

By *(5.5)*, $v_{i,i} = 0$ for incompressible flow. Here $v_{1,1} = A[-4x_1(x_1^2 - x_2^2)/r^6 + 2x_1/r^4]$ and $v_{2,2} = A[2x_1/r^4 - 8x_1x_2^2/r^6]$. Adding, $v_{1,1} + v_{2,2} = 0$.

5.23. Show that the flow of Problem 5.22 is irrotational.

By *(4.29)*, curl v = 0 for irrotational flow. Thus

$$\text{curl}\,\mathbf{v} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ A(x_1^2 - x_2^2)/r^4 & 2Ax_1x_2/r^4 & 0 \end{vmatrix}$$

$$= A[2x_2/r^4 - 8x_1^2x_2/r^6 + 2x_2/r^4 + 4x_2(x_1^2 - x_2^2)/r^6]\,\hat{\mathbf{e}}_3 = 0$$

5.24. In a two dimensional incompressible steady flow, $v_1 = -Ax_2/r^2$ where $r^2 = x_1^2 + x_2^2$. Determine v_2 if $v_2 = 0$ at $x_1 = 0$ for all x_2. Show that the motion is irrotational and that the streamlines are circles.

From *(5.5)*, $v_{i,i} = 0$ or $v_{1,1} = -v_{2,2} = 2Ax_1x_2/r^4$ for this incompressible flow. Integrating with respect to x_2 and imposing the given conditions on v_2 yields $v_2 = Ax_1/r^2$.

For an irrotational motion, curl v = 0. Here

$$\text{curl}\,\mathbf{v} = A[(x_1^2 - x_2^2)/r^4 + (-x_1^2 + x_2^2)/r^4]\,\hat{\mathbf{e}}_3 = 0$$

From Problem 4.7, page 118, the equations for streamlines are $dx_1/v_1 = dx_2/v_2$. Here these equations are $x_1\,dx_1 + x_2\,dx_2 = 0$ which integrate directly into the circles $x_1^2 + x_2^2 = \text{constant}$.

5.25. For a continuum whose constitutive equations are $\sigma_{ij} = (-p + \lambda^* D_{kk})\delta_{ij} + 2\mu^* D_{ij}$, determine the equations of motion in terms of the velocity v_i.

From *(5.16)*, $\rho\dot{v}_i = \rho b_i + \sigma_{ij,j}$ or here $\rho\dot{v}_i = \rho b_i - p_{,j}\delta_{ij} + \lambda^* D_{kk,j}\delta_{ij} + 2\mu^* D_{ij,j}$. By definition, $2D_{ij} = v_{i,j} + v_{j,i}$ so that $D_{kk} = v_{k,k}$ and $2D_{ij,j} = v_{i,jj} + v_{j,ij}$. Therefore

$$\rho\dot{v}_i = \rho b_i - p_{,i} + (\lambda^* + \mu^*)v_{j,ij} + \mu^* v_{i,jj}$$

In symbolic notation this equation is

$$\rho\dot{\mathbf{v}} = \rho\mathbf{b} - \nabla p + (\lambda^* + \mu^*)\nabla(\nabla\cdot\mathbf{v}) + \mu^*\nabla^2\mathbf{v}$$

5.26. If the continuum of Problem 5.25 is considered incompressible, show that the divergence of the vorticity vanishes and give the form of the equations of motion for this case.

By (*4.29*), $q_i = \epsilon_{ijk}v_{k,j}$; and $\operatorname{div}\mathbf{q} = \epsilon_{ijk}v_{k,ji} = 0$ since ϵ_{ijk} is antisymmetric and $v_{k,ji}$ is symmetric in i and j. Thus for $\nabla\cdot\mathbf{v} = 0$ the equations of motion become $\rho\dot{v}_i = \rho b_i - p_{,i} + \mu^* v_{i,jj}$ or in Gibbs notation $\rho\dot{\mathbf{v}} = \rho\mathbf{b} - \nabla p + \mu^*\nabla^2\mathbf{v}$.

5.27. Determine the material rate of change of the kinetic energy of the continuum which occupies the volume V and give the meaning of the resulting integrals.

By (*5.23*), $dK/dt = \int_V \rho v_i \dot{v}_i \, dV$. Also the total stress power of the surface forces is $\int_S v_i t_i^{(\hat{n})} \, dS$ which may be written $\int_S v_i \sigma_{ij} n_j \, dS$ and by the divergence theorem (*1.157*) and the equations of motion (*5.16*) expressed as the volume integrals $\int_S v_i \sigma_{ij} n_j \, dS = \int_V \sigma_{ij} v_{i,j} \, dV + \int_V \rho(v_i \dot{v}_i - b_i v_i) \, dV$. Thus

$$\frac{dK}{dt} = \int_V \rho b_i v_i \, dV - \int_V \sigma_{ij} v_{i,j} \, dV + \int_S v_i t_i^{(\hat{n})} \, dS$$

This sum of integrals represents the rate of work done by the body forces, the internal stresses and the surface stresses, respectively.

5.28. A continuum for which $\sigma_{ij}^{(D)} = \lambda^* D_{kk}\delta_{ij} + 2\mu^* D_{ij}$ undergoes an incompressible irrotational flow with a velocity potential ϕ such that $\mathbf{v} = \operatorname{grad}\phi$. Determine the dissipation function $\sigma_{ij}^{(D)}\dot{\epsilon}_{ij}$.

Here $\sigma_{ij}^{(D)}\dot{\epsilon}_{ij} = \sigma_{ij}^{(D)} D_{ij} = (\lambda^* D_{kk}\delta_{ij} + 2\mu^* D_{ij})D_{ij} = 2\mu^* D_{ij}D_{ij}$ since $D_{kk} = v_{k,k} = 0$ for incompressible flow. Also since $v_i = \phi_{,i}$, the scalar $D_{ij}D_{ij} = \phi_{,ij}\phi_{,ij}$ and so $\sigma_{ij}^{(D)} D_{ij} = 2\mu^*\phi_{,ij}\phi_{,ij}$.

Because the motion is incompressible and irrotational, $\phi_{,ii} = 0$ and $2\phi_{,ij}\phi_{,ij} = (\phi_{,i}\phi_{,i})_{,jj} = \nabla^2(\nabla\phi)^2$. It is also interesting to note that

$$\nabla^4(\phi^2) = (\phi\phi)_{,iijj} = 2(\phi_{,iijj}\phi + 4\phi_{,ijj}\phi_{,i} + \phi_{,ii}\phi_{,jj} + 2\phi_{,ij}\phi_{,ij})$$

which for $\phi_{,ii} = 0$ reduces to $4\phi_{,ij}\phi_{,ij}$. Thus $\phi_{ij}^{(D)}\dot{\epsilon}_{ij} = \mu^*\nabla^2(\nabla\phi)^2 = \mu^*\nabla^4(\phi^2)/2$.

5.29. For a continuum with $\sigma_{ij} = -p\delta_{ij}$, the *specific enthalpy* $h = u + p/\rho$. Show that the energy equation may be written $\dot{h} = \dot{p}/\rho + T\dot{s}$ using this definition for the enthalpy.

From (*5.41*), $\dot{u} = -p\delta_{ij}D_{ij}/\rho + T\dot{s}$ for the given stress; and by the result of Problem 5.13 and the definition of h, $\dot{u} = \dot{h} - \dot{p}/\rho - p\dot{\rho}/\rho^2 = -p\dot{\rho}/\rho^2 + T\dot{s}$. Canceling and rearranging, $\dot{h} = \dot{p}/\rho + T\dot{s}$.

5.30. If the continuum of Problem 5.25 undergoes an incompressible flow, determine the equation of motion in terms of the vorticity $\mathbf{q}$ in the absence of body forces and assuming constant density.

For incompressible flow, $\nabla\cdot\mathbf{v} = 0$; and if $\mathbf{b} \equiv 0$, the equation of motion in Problem 5.25 reduces to $\rho\dot{v}_i = -p_{,i} + \mu^* v_{i,jj}$. Taking the cross product $\nabla\times$ with this equation for $\rho = \text{constant}$ gives $\epsilon_{pqi}\dot{v}_{i,q} = -\epsilon_{pqi}p_{,iq}/\rho + (\mu^*/\rho)\epsilon_{pqi}v_{i,jjq}$. But $\epsilon_{pqi}p_{,iq} = 0$ and by (*4.29*) the result is $\dot{q}_p = (\mu^*/\rho)q_{p,jj}$. In symbolic notation, $d\mathbf{q}/dt = (\mu^*/\rho)\nabla^2\mathbf{q}$.

Supplementary Problems

5.31. Show that for the rate of rotation vector $\mathbf{\Omega}$, $\dfrac{d}{dt}\left(\dfrac{\mathbf{\Omega}}{\rho}\right) = \dfrac{\mathbf{\Omega}\cdot\nabla\mathbf{v}}{\rho}$.

5.32. Show that the flow represented by $v_1 = -2x_1x_2x_3/r^4$, $v_2 = (x_1^2 - x_2^2)x_3/r^4$, $v_3 = x_2/r^2$ where $r^2 = x_1^2 + x_2^2$, satisfies conditions for an incompressible flow. Is this motion irrotational?

5.33. In terms of Cartesian coordinates x, y, z the continuity equation is

$$\partial\rho/\partial t + \partial(\rho v_x)/\partial x + \partial(\rho v_y)/\partial y + \partial(\rho v_z)/\partial z = 0$$

Show that in terms of cylindrical coordinates r, θ, z this equation becomes

$$r(\partial\rho/\partial t) + \partial(r\rho v_r)/\partial r + \partial(\rho v_\theta)/\partial\theta + r(\partial(\rho v_z)/\partial z) = 0$$

5.34. Show that the flow $v_r = (1-r^2)\cos\theta/r^2$, $v_\theta = (1+r^2)\sin\theta/r^2$, $v_z = 0$ satisfies the continuity equation in cylindrical coordinates when the density ρ is a constant.

5.35. If $P_{ij\ldots}(\mathbf{x}, t)$ is an arbitrary scalar, vector or tensor function, show that

$$\int_S P_{ij\ldots}\sigma_{pq}n_q\,dS = \int_V [\sigma_{pq}P_{ij\ldots,q} + \rho P_{ij\ldots}(\dot{v}_p - b_p)]\,dV$$

5.36. If a continuum is subjected to a body moment per unit mass $\mathbf{h}$ in addition to body force $\mathbf{b}$, and a couple stress $\mathbf{g}^{(\hat{\mathbf{n}})}$ in addition to the stress $\mathbf{t}^{(\hat{\mathbf{n}})}$, the angular momentum balance may be written

$$\frac{d}{dt}\int_V \rho(\mathbf{m} + \mathbf{x}\times\mathbf{v})\,dV = \int_V (\mathbf{h} + \mathbf{x}\times\mathbf{b})\,dV + \int_S (\mathbf{g}^{(\hat{\mathbf{n}})} + \mathbf{x}\times\mathbf{t}^{(\hat{\mathbf{n}})})\,dS$$

where $\mathbf{m}$ is distributed angular momentum per unit mass. If $\hat{\mathbf{n}}\cdot\mathbf{G} = \mathbf{g}^{(\hat{\mathbf{n}})}$, show that the local form of this relation is $\rho\,d\mathbf{m}/dt = \mathbf{h} + \nabla\cdot\mathbf{G} + \mathbf{\Sigma}_v$.

5.37. If a continuum has the constitutive equation $\sigma_{ij} = -p\delta_{ij} + \beta D_{ij} + \alpha D_{ik}D_{kj}$, show that $\sigma_{ii} = 3(-p - 2\alpha \mathrm{II}_D/3)$. Assume incompressibility, $D_{ii} = 0$.

5.38. For a continuum having $\sigma_{ij} = -p\delta_{ij}$, show that $du = T\,ds - p\,dv$ where *in this problem* $v = 1/\rho$, the specific volume.

5.39. If $T\,ds/dt = -v_{i,i}/\rho$ and the specific free energy is defined by $\Psi = u - Ts$, show that the energy equation may be written $\rho\,d\Psi/dt + \rho s\,dT/dt = \sigma_{ij}D_{ij}$.

5.40. For a thermomechanical continuum having the constitutive equation

$$\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} - (3\lambda + 2\mu)\alpha\,\delta_{ij}(T - T_0)$$

where T_0 is a reference temperature, show that $\epsilon_{kk} = 3\alpha(T - T_0)$ when $\sigma_{ij} = s_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$.

Chapter 6

Linear Elasticity

6.1 GENERALIZED HOOKE'S LAW. STRAIN ENERGY FUNCTION

In classical linear elasticity theory it is assumed that *displacements* and *displacement gradients* are sufficiently small that no distinction need be made between the Lagrangian and Eulerian descriptions. Accordingly in terms of the displacement vector u_i, the linear strain tensor is given by the equivalent expressions

$$l_{ij} \;=\; \epsilon_{ij} \;=\; \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \;=\; \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \;=\; \tfrac{1}{2}(u_{i,j} + u_{j,i})$$

or

$$\mathbf{L} = \mathbf{E} = \tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{X}} + \nabla_{\mathbf{X}}\mathbf{u}) = \tfrac{1}{2}(\mathbf{u}\nabla_{\mathbf{x}} + \nabla_{\mathbf{x}}\mathbf{u}) = \tfrac{1}{2}(\mathbf{u}\nabla + \nabla\mathbf{u}) \tag{6.1}$$

In the following it is further assumed that the deformation processes are *adiabatic* (no heat loss or gain) and *isothermal* (constant temperature) unless specifically stated otherwise.

The constitutive equations for a linear elastic solid relate the stress and strain tensors through the expression

$$\sigma_{ij} = C_{ijkm}\epsilon_{km} \qquad \text{or} \qquad \mathbf{\Sigma} = \tilde{\mathbf{C}}:\mathbf{E} \tag{6.2}$$

which is known as the *generalized Hooke's law.* In (*6.2*) the *tensor* of *elastic constants* C_{ijkm} has 81 components. However, due to the symmetry of both the stress and strain tensors, there are at most 36 distinct elastic constants. For the purpose of writing Hooke's law in terms of these 36 components, the double indexed system of stress and strain components is often replaced by a single indexed system having a range of 6. Thus in the notation

$$\begin{array}{ll} \sigma_{11} = \sigma_1 & \sigma_{23} = \sigma_{32} = \sigma_4 \\ \sigma_{22} = \sigma_2 & \sigma_{13} = \sigma_{31} = \sigma_5 \\ \sigma_{33} = \sigma_3 & \sigma_{12} = \sigma_{21} = \sigma_6 \end{array} \tag{6.3}$$

and

$$\begin{array}{ll} \epsilon_{11} = \epsilon_1 & 2\epsilon_{23} = 2\epsilon_{32} = \epsilon_4 \\ \epsilon_{22} = \epsilon_2 & 2\epsilon_{13} = 2\epsilon_{31} = \epsilon_5 \\ \epsilon_{33} = \epsilon_3 & 2\epsilon_{12} = 2\epsilon_{21} = \epsilon_6 \end{array} \tag{6.4}$$

Hooke's law may be written

$$\sigma_K = C_{KM}\epsilon_M \quad (K,M = 1,2,3,4,5,6) \tag{6.5}$$

where C_{KM} represents the 36 elastic constants, and where upper case Latin subscripts are used to emphasize the range of 6 on these indices.

When thermal effects are neglected, the energy balance equation (*5.32*) may be written

$$\frac{du}{dt} = \frac{1}{\rho}\sigma_{ij}D_{ij} = \frac{1}{\rho}\sigma_{ij}\dot{\epsilon}_{ij} \tag{6.6}$$

The internal energy in this case is purely mechanical and is called the *strain energy* (per unit mass). From (*6.6*),

$$du = \frac{1}{\rho}\sigma_{ij}\,d\epsilon_{ij} \tag{6.7}$$

and if u is considered a function of the nine strain components, $u = u(\epsilon_{ij})$, its differential is given by

$$du = \frac{\partial u}{\partial \epsilon_{ij}}\,d\epsilon_{ij} \tag{6.8}$$

Comparing (*6.7*) and (*6.8*), it is observed that

$$\frac{1}{\rho}\sigma_{ij} = \frac{\partial u}{\partial \epsilon_{ij}} \tag{6.9}$$

The *strain energy density* u^* (per unit volume) is defined as

$$u^* = \rho u \tag{6.10}$$

and since ρ may be considered a constant in the small strain theory, u^* has the property that

$$\sigma_{ij} = \rho\frac{\partial u}{\partial \epsilon_{ij}} = \frac{\partial u^*}{\partial \epsilon_{ij}} \tag{6.11}$$

Furthermore, the zero state of strain energy may be chosen arbitrarily; and since the stress must vanish with the strains, the simplest form of strain energy function that leads to a linear stress-strain relation is the quadratic form

$$u^* = \tfrac{1}{2}C_{ijkm}\epsilon_{ij}\epsilon_{km} \tag{6.12}$$

From (*6.2*), this equation may be written

$$u^* = \tfrac{1}{2}\sigma_{ij}\epsilon_{ij} \qquad \text{or} \qquad u^* = \tfrac{1}{2}\mathbf{\Sigma} : \mathbf{E} \tag{6.13}$$

In the single indexed system of symbols, (*6.12*) becomes

$$u^* = \tfrac{1}{2}C_{KM}\epsilon_K\epsilon_M \tag{6.14}$$

in which $C_{KM} = C_{MK}$. Because of this symmetry on C_{KM}, the number of independent elastic constants is at most 21 if a strain energy function exists.

6.2 ISOTROPY. ANISOTROPY. ELASTIC SYMMETRY

If the elastic properties are independent of the reference system used to describe it, a material is said to be *elastically isotropic*. A material that is not isotropic is called *anisotropic*. Since the elastic properties of a Hookean solid are expressed through the coefficients C_{KM}, a general anisotropic body will have an *elastic-constant matrix* of the form

$$[C_{KM}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \tag{6.15}$$

When a strain energy function exists for the body, $C_{KM} = C_{MK}$, and the 36 constants in (*6.15*) are reduced to 21.

A *plane of elastic symmetry* exists at a point where the elastic constants have the same values for every pair of coordinate systems which are the reflected images of one another with respect to the plane. The axes of such coordinate systems are referred to as "equivalent elastic directions." If the x_1x_2 plane is one of elastic symmetry, the constants C_{KM} are invariant under the coordinate transformation

$$x_1' = x_1, \quad x_2' = x_2, \quad x_3' = -x_3 \qquad (6.16)$$

as shown in Fig. 6-1. The transformation matrix of (*6.16*) is given by

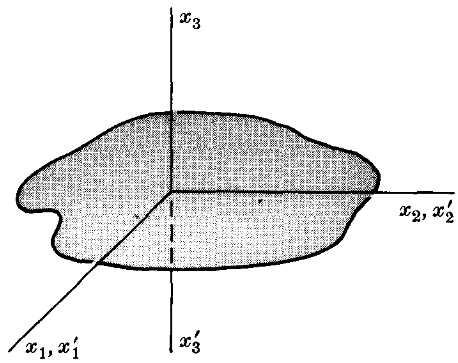


Fig. 6-1

$$[a_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad (6.17)$$

Inserting the values of (*6.17*) into the transformation laws for the linear stress and strain tensors, (*2.27*) and (*3.78*) respectively, the elastic matrix for a material having x_1x_2 as a plane of symmetry is

$$[C_{KM}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{21} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{31} & C_{32} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{54} & C_{55} & 0 \\ C_{61} & C_{62} & C_{63} & 0 & 0 & C_{66} \end{bmatrix} \qquad (6.18)$$

The 20 constants in (*6.18*) are reduced to 13 when a strain energy function exists.

If a material possesses three mutually perpendicular planes of elastic symmetry, the material is called *orthotropic* and its elastic matrix is of the form

$$[C_{KM}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \qquad (6.19)$$

having 12 independent constants, or 9 if $C_{KM} = C_{MK}$.

An *axis of elastic symmetry* of order N exists at a point when there are sets of equivalent elastic directions which can be superimposed by a rotation through an angle of $2\pi/N$ about the axis. Certain cases of axial and plane elastic symmetry are equivalent.

6.3 ISOTROPIC MEDIA. ELASTIC CONSTANTS

Bodies which are elastically equivalent in all directions possess complete symmetry and are termed *isotropic*. Every plane and every axis is one of elastic symmetry in this case.

For isotropy, the number of independent elastic constants reduces to 2, and the elastic matrix is symmetric regardless of the existence of a strain energy function. Choosing as the two independent constants the well-known Lamé constants, λ and μ, the matrix (*6.19*) reduces to the isotropic elastic form

$$[C_{KM}] = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \qquad (6.20)$$

In terms of λ and μ, Hooke's law (*6.2*) for an isotropic body is written

$$\sigma_{ij} = \lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij} \quad \text{or} \quad \mathbf{\Sigma} = \lambda\mathbf{I}\epsilon + 2\mu\mathbf{E} \qquad (6.21)$$

where $\epsilon = \epsilon_{kk} = \mathrm{I}_{\mathbf{E}}$. This equation may be readily inverted to express the strains in terms of the stresses as

$$\epsilon_{ij} = \frac{-\lambda}{2\mu(3\lambda+2\mu)}\delta_{ij}\sigma_{kk} + \frac{1}{2\mu}\sigma_{ij} \quad \text{or} \quad \mathbf{E} = \frac{-\lambda}{2\mu(3\lambda+2\mu)}\mathbf{I}\Theta + \frac{1}{2\mu}\mathbf{\Sigma} \qquad (6.22)$$

where $\Theta = \sigma_{kk} = \mathrm{I}_{\Sigma}$, the symbol traditionally used in elasticity for the first stress invariant.

For a simple uniaxial state of stress in the x_1 direction, engineering constants E and ν may be introduced through the relationships $\sigma_{11} = E\epsilon_{11}$ and $\epsilon_{22} = \epsilon_{33} = -\nu\epsilon_{11}$. The constant E is known as *Young's modulus*, and ν is called *Poisson's ratio*. In terms of these elastic constants Hooke's law for isotropic bodies becomes

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\epsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\epsilon_{kk}\right) \quad \text{or} \quad \mathbf{\Sigma} = \frac{E}{1+\nu}\left(\mathbf{E} + \frac{\nu}{1-2\nu}\mathbf{I}\epsilon\right) \qquad (6.23)$$

or, when inverted,

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk} \quad \text{or} \quad \mathbf{E} = \frac{1+\nu}{E}\mathbf{\Sigma} - \frac{\nu}{E}\mathbf{I}\Theta \qquad (6.24)$$

From a consideration of a uniform hydrostatic pressure state of stress, it is possible to define the *bulk modulus*,

$$K = \frac{E}{3(1-2\nu)} \quad \text{or} \quad K = \frac{3\lambda+2\mu}{3} \qquad (6.25)$$

which relates the pressure to the cubical dilatation of a body so loaded. For a so-called state of pure shear, the *shear modulus* G relates the shear components of stress and strain. G is actually equal to μ and the expression

$$\mu = G = \frac{E}{2(1+\nu)} \qquad (6.26)$$

may be proven without difficulty.

6.4 ELASTOSTATIC PROBLEMS. ELASTODYNAMIC PROBLEMS

In an elastostatic problem of a homogeneous isotropic body, certain field equations, namely,

(*a*) Equilibrium equations,

$$\sigma_{ji,j} + \rho b_i = 0 \quad \text{or} \quad \nabla\cdot\mathbf{\Sigma} + \rho\mathbf{b} = 0 \qquad (6.27)$$

(b) Hooke's law,

$$\sigma_{ij} = \lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij} \qquad \text{or} \qquad \mathbf{\Sigma} = \lambda\mathbf{I}\epsilon + 2\mu\mathbf{E} \tag{6.28}$$

(c) Strain-displacement relations,

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \qquad \text{or} \qquad \mathbf{E} = \tfrac{1}{2}(\mathbf{u}\nabla + \nabla\mathbf{u}) \tag{6.29}$$

must be satisfied at all interior points of the body. Also, prescribed conditions on stress and/or displacements must be satisfied on the bounding surface of the body.

The boundary value problems of elasticity are usually classified according to boundary conditions into problems for which

(1) displacements are prescribed everywhere on the boundary,

(2) stresses (surface tractions) are prescribed everywhere on the boundary,

(3) displacements are prescribed over a portion of the boundary, stresses are prescribed over the remaining part.

For all three categories the body forces are assumed to be given throughout the continuum.

For those problems in which boundary displacement components are given everywhere by an equation of the form

$$u_i = g_i(\mathbf{X}) \qquad \text{or} \qquad \mathbf{u} = \mathbf{g}(\mathbf{X}) \tag{6.30}$$

the strain-displacement relations (*6.29*) may be substituted into Hooke's law (*6.28*) and the result in turn substituted into (*6.27*) to produce the governing equations,

$$\mu u_{i,jj} + (\lambda+\mu)u_{j,ji} + \rho b_i = 0 \qquad \text{or} \qquad \mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} + \rho\mathbf{b} = 0 \tag{6.31}$$

which are called the *Navier-Cauchy* equations. The solution of this type of problem is therefore given in the form of the displacement vector u_i, satisfying (*6.31*) throughout the continuum and fulfilling (*6.30*) on the boundary.

For those problems in which surface tractions are prescribed everywhere on the boundary by equations of the form

$$t_i^{(\hat{n})} = \sigma_{ij}n_j \qquad \text{or} \qquad \mathbf{t}^{(\hat{n})} = \mathbf{\Sigma}\cdot\hat{\mathbf{n}} \tag{6.32}$$

the equations of compatibility (*3.104*) may be combined with Hooke's law (*6.24*) and the equilibrium equation (*6.27*) to produce the governing equations,

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} + \rho(b_{i,j} + b_{j,i}) + \frac{\nu}{1-\nu}\delta_{ij}\rho b_{k,k} = 0$$

or

$$\nabla^2\mathbf{\Sigma} + \frac{1}{1+\nu}\nabla\nabla\Theta + \rho(\nabla\mathbf{b} + \mathbf{b}\nabla) + \frac{\nu}{1-\nu}\mathbf{I}\rho\nabla\cdot\mathbf{b} = 0 \tag{6.33}$$

which are called the *Beltrami-Michell* equations of compatibility. The solution for this type of problem is given by specifying the stress tensor which satisfies (*6.33*) throughout the continuum and fulfills (*6.32*) on the boundary.

For those problems having "mixed" boundary conditions, the system of equations (*6.27*), (*6.28*) and (*6.29*) must be solved. The solution gives the stress and displacement fields throughout the continuum. The stress components must satisfy (*6.32*) over some portion of the boundary, while the displacements satisfy (*6.30*) over the remainder of the boundary.

In the formulation of elastodynamics problems, the equilibrium equations (*6.27*) must be replaced by the equations of motion (*5.16*)

$$\sigma_{ij,j} + \rho b_i = \rho\dot{v}_i \qquad \text{or} \qquad \nabla\cdot\mathbf{\Sigma} + \rho\mathbf{b} = \rho\dot{\mathbf{v}} \tag{6.34}$$

and *initial conditions* as well as *boundary conditions* must be specified. In terms of the displacement field u_i, the governing equation here, analogous to (*6.31*) in the elastostatic case is

$$\mu u_{i,jj} + (\lambda+\mu)u_{j,ji} + \rho b_i = \rho \ddot{u}_i \qquad \text{or} \qquad \mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} + \rho\mathbf{b} = \rho\ddot{\mathbf{u}} \tag{6.35}$$

Solutions of (*6.35*) appear in the form $u_i = u_i(\mathbf{x}, t)$ and must satisfy not only initial conditions on the motion, usually expressed by equations such as

$$u_i = u_i(\mathbf{x}, 0) \qquad \text{and} \qquad \dot{u}_i = \dot{u}_i(\mathbf{x}, 0) \tag{6.36}$$

but also boundary conditions, either on the displacements,

$$u_i = g_i(\mathbf{x}, t) \qquad \text{or} \qquad \mathbf{u} = \mathbf{g}(\mathbf{x}, t) \tag{6.37}$$

or on the surface tractions,

$$t_i^{(\hat{\mathbf{n}})} = t_i^{(\hat{\mathbf{n}})}(\mathbf{x}, t) \qquad \text{or} \qquad \mathbf{t}^{(\hat{\mathbf{n}})} = \mathbf{t}^{(\hat{\mathbf{n}})}(\mathbf{x}, t) \tag{6.38}$$

6.5 THEOREM OF SUPERPOSITION. UNIQUENESS OF SOLUTIONS. ST. VENANT PRINCIPLE

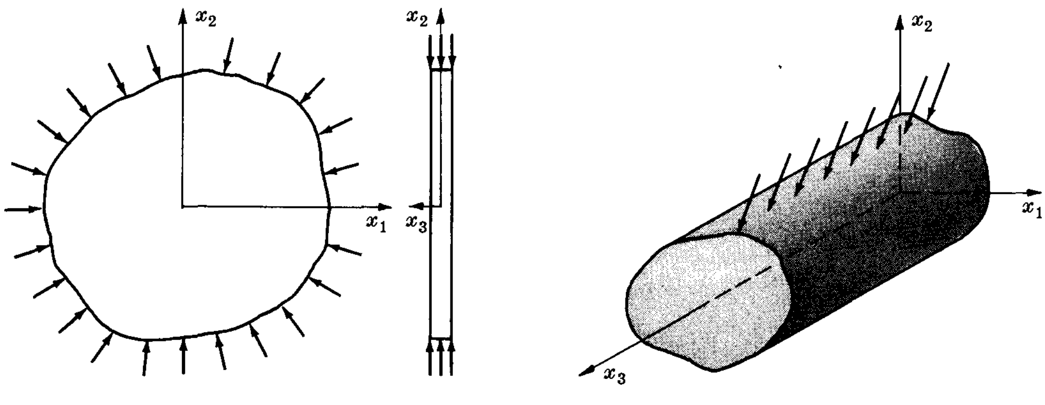
Because the equations of linear elasticity are linear equations, the principle of superposition may be used to obtain additional solutions from those previously established. If, for example, $\sigma_{ij}^{(1)}, u_i^{(1)}$ represent a solution to the system (*6.27*), (*6.28*) and (*6.29*) with body forces $b_i^{(1)}$, and $\sigma_{ij}^{(2)}, u_i^{(2)}$ represent a solution for body forces $b_i^{(2)}$, then $\sigma_{ij} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}$, $u_i = u_i^{(1)} + u_i^{(2)}$ represent a solution to the system for body forces $b_i = b_i^{(1)} + b_i^{(2)}$.

The uniqueness of a solution to the general elastostatic problem of elasticity may be established by use of the superposition principle, together with the law of conservation of energy. A proof of uniqueness is included among the exercises that follow.

St. Venant's principle is a statement regarding the differences that occur in the stresses and strains at some interior location of an elastic body, due to two separate but statically equivalent systems of surface tractions, being applied to some portion of the boundary. The principle asserts that, for locations sufficiently remote from the area of application of the loadings, the differences are negligible. This assumption is often of great assistance in solving practical problems.

6.6 TWO-DIMENSIONAL ELASTICITY. PLANE STRESS AND PLANE STRAIN

Many problems in elasticity may be treated satisfactorily by a two-dimensional, or *plane theory of elasticity*. There are two general types of problems involved in this plane analysis. Although these two types may be defined by setting down certain restrictions and assumptions on the stress and displacement fields, they are often introduced descriptively in terms of their physical prototypes. In *plane stress* problems, the geometry of the body is essentially that of a plate with one dimension much smaller than the others. The loads are applied uniformly over the thickness of the plate and act in the plane of the plate as shown in Fig. 6-2(*a*) below. In *plane strain* problems, the geometry of the body is essentially that of a prismatic cylinder with one dimension much larger than the others. The loads are uniformly distributed with respect to the large dimension and act perpendicular to it as shown in Fig. 6-2(*b*) below.



(a) (b)

Fig. 6-2

For the plane stress problem of Fig. 6-2(a) the stress components $\sigma_{33}, \sigma_{13}, \sigma_{23}$ are taken as zero everywhere, and the remaining components are taken as functions of x_1 and x_2 only,

$$\sigma_{\alpha\beta} = \sigma_{\alpha\beta}(x_1, x_2) \qquad (\alpha, \beta = 1, 2) \tag{6.39}$$

Accordingly, the field equations for plane stress are

(a) $$\sigma_{\alpha\beta,\beta} + \rho b_\alpha = 0 \quad \text{or} \quad \nabla \cdot \Sigma + \rho \mathbf{b} = 0 \tag{6.40}$$

(b) $$\epsilon_{\alpha\beta} = \frac{1+\nu}{E}\sigma_{\alpha\beta} - \frac{\nu}{E}\delta_{\alpha\beta}\sigma_{\gamma\gamma} \quad \text{or} \quad \mathbf{E} = \frac{1+\nu}{E}\Sigma - \frac{\nu}{E}\mathbf{I}\Theta$$
$$\epsilon_{33} = -\frac{\nu}{E}\sigma_{\alpha\alpha}\,. \tag{6.41}$$

(c) $$\epsilon_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \quad \text{or} \quad \mathbf{E} = \tfrac{1}{2}(\mathbf{u}\nabla + \nabla\mathbf{u}) \tag{6.42}$$

in which $\nabla \equiv \frac{\partial}{\partial x_1}\hat{\mathbf{e}}_1 + \frac{\partial}{\partial x_2}\hat{\mathbf{e}}_2$ and

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{E} = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{pmatrix} \tag{6.43}$$

Due to the particular form of the strain tensor in the plane stress case, the six compatibility equations (*3.104*) may be reduced with reasonable accuracy for very thin plates to the single equation

$$\epsilon_{11,22} + \epsilon_{22,11} = 2\epsilon_{12,12} \tag{6.44}$$

In terms of the displacement components u_α, the field equations may be combined to give the governing equation

$$\frac{E}{2(1+\nu)}\nabla^2 u_\alpha + \frac{E}{2(1-\nu)}u_{\beta,\beta\alpha} + \rho b_\alpha = 0 \quad \text{or} \quad \frac{E}{2(1+\nu)}\nabla^2\mathbf{u} + \frac{E}{2(1-\nu)}\nabla\nabla\cdot\mathbf{u} + \rho\mathbf{b} = 0 \tag{6.45}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$.

For the plane strain problem of Fig. 6-2(b) the displacement component u_3 is taken as zero, and the remaining components considered as functions of x_1 and x_2 only,

$$u_\alpha = u_\alpha(x_1, x_2) \tag{6.46}$$

In this case, the field equations may be written

$$(a) \qquad \sigma_{\alpha\beta,\beta} + \rho b_\alpha = 0 \qquad \text{or} \qquad \nabla \cdot \Sigma + \rho \mathbf{b} = 0 \tag{6.47}$$

$$(b) \qquad \sigma_{\alpha\beta} = \lambda \delta_{\alpha\beta} \epsilon_{\gamma\gamma} + 2\mu\epsilon_{\alpha\beta} \qquad \text{or} \qquad \Sigma = \lambda \mathbf{I}\epsilon + 2\mu \mathbf{E}$$
$$\sigma_{33} = \nu\sigma_{\alpha\alpha} = \frac{\lambda}{2(\lambda+\mu)}\sigma_{\alpha\alpha} \tag{6.48}$$

$$(c) \qquad \epsilon_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \qquad \text{or} \qquad \mathbf{E} = \tfrac{1}{2}(\mathbf{u}\nabla + \nabla\mathbf{u}) \tag{6.49}$$

in which
$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \qquad \text{and} \qquad \mathbf{E} = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.50}$$

From (*6.47*), (*6.48*), (*6.49*), the appropriate Navier equation for plane strain is

$$\mu\nabla^2 u_\alpha + (\lambda+\mu)u_{\beta,\beta\alpha} + \rho b_\alpha = 0 \qquad \text{or} \qquad \mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} + \rho\mathbf{b} = 0 \tag{6.51}$$

As in the case of plane stress, the compatibility equations for plane strain reduce to the single equation (*6.44*).

If the forces applied to the edge of the plate in Fig. 6-2(*a*) are not uniform across the thickness, but are symmetrical with respect to the middle plane of the plate, a state of *generalized plane stress* is said to exist. In formulating problems for this case, the field variables $\sigma_{\alpha\beta}$, $\epsilon_{\alpha\beta}$ and u_α must be replaced by stress, strain and displacement variables averaged across the thickness of the plate. In terms of such averaged field variables, the generalized plane stress formulation is essentially the same as the plane strain case if λ is replaced by

$$\lambda' = \frac{2\lambda\mu}{\lambda + 2\mu} = \frac{\nu E}{1-\nu^2} \tag{6.52}$$

A case of *generalized plane strain* is sometimes mentioned in elasticity books when ϵ_{33} is taken as a constant other than zero in (*6.50*).

6.7 AIRY'S STRESS FUNCTION

If body forces are absent or are constant, the solution of *plane elastostatic problems* (plane strain or generalized plane stress problems) is often obtained through the use of the *Airy stress function.* Even if body forces must be taken into account, the superposition principle allows for their contribution to the solution to be introduced as a particular integral of the linear differential field equations.

For plane elastostatic problems in the absence of body forces, the equilibrium equations reduce to

$$\sigma_{\alpha\beta,\beta} = 0 \qquad \text{or} \qquad \nabla\cdot\Sigma = 0 \tag{6.53}$$

and the compatibility equation (*6.44*) may be expressed in terms of stress components as

$$\nabla^2(\sigma_{11}+\sigma_{22}) = 0, \qquad \nabla^2\Theta_1 = 0 \tag{6.54}$$

The stress components are now given as partial derivatives of the Airy stress function $\phi = \phi(x_1, x_2)$ in accordance with the equations

$$\sigma_{11} = \phi_{,22}, \qquad \sigma_{12} = -\phi_{,12}, \qquad \sigma_{22} = \phi_{,11} \tag{6.55}$$

The equilibrium equations (*6.53*) are satisfied identically, and the compatibility condition (*6.54*) becomes the *biharmonic equation*

$$\nabla^2(\nabla^2\phi) \;=\; \nabla^4\phi \;=\; \phi_{,1111} + 2\phi_{,1122} + \phi_{,2222} \;=\; 0 \tag{6.56}$$

Functions which satisfy (*6.56*) are called *biharmonic functions*. By considering biharmonic functions with single-valued second partial derivatives, numerous solutions to plane elastostatic problems may be constructed, which satisfy automatically both equilibrium and compatibility. Of course these solutions must be tailored to fit whatever boundary conditions are prescribed.

6.8 TWO-DIMENSIONAL ELASTOSTATIC PROBLEMS IN POLAR COORDINATES

Body geometry often deems it convenient to formulate two-dimensional elastostatic problems in terms of polar coordinates r and θ. Thus for transformation equations

$$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta \tag{6.57}$$

the stress components shown in Fig. 6-3 are found to lead to equilibrium equations in the form

$$\frac{\partial\sigma_{(rr)}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{(r\theta)}}{\partial\theta} + \frac{\sigma_{(rr)} - \sigma_{(\theta\theta)}}{r} + R \;=\; 0 \tag{6.58}$$

$$\frac{1}{r}\frac{\partial\sigma_{(\theta\theta)}}{\partial\theta} + \frac{\partial\sigma_{(r\theta)}}{\partial r} + \frac{2\sigma_{(r\theta)}}{r} + Q \;=\; 0 \tag{6.59}$$

in which R and Q represent body forces per unit volume in the directions shown.

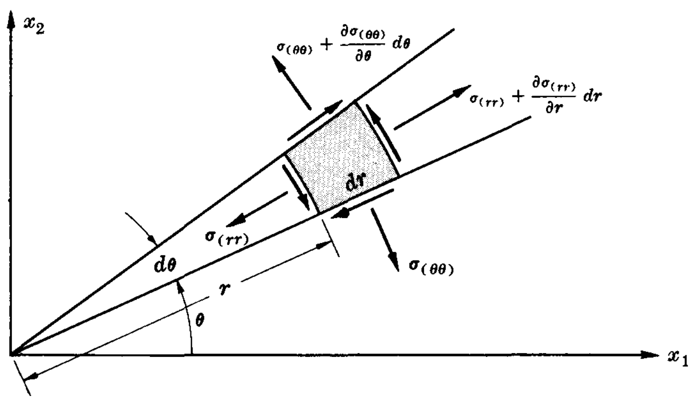


Fig. 6-3

Taking the Airy stress function now as $\Phi = \Phi(r,\theta)$, the stress components are given by

$$\sigma_{(rr)} \;=\; \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} \tag{6.60}$$

$$\sigma_{(\theta\theta)} \;=\; \partial^2\Phi/\partial r^2 \tag{6.61}$$

$$\sigma_{(r\theta)} \;=\; -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right) \tag{6.62}$$

The compatibility condition again leads to the biharmonic equation

$$\nabla^2(\nabla^2\Phi) \;=\; \nabla^4\Phi \;=\; 0 \tag{6.63}$$

but, in polar form, $\nabla^2 \;=\; \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}$.

6.9 HYPERELASTICITY. HYPOELASTICITY

Modern continuum studies have led to constitutive equations which define materials that are elastic in a special sense. In this regard a material is said to be *hyperelastic* if it possesses a strain energy function U such that the material derivative of this function is equal to the stress power per unit volume. Thus the constitutive equation is of the form

$$\frac{d}{dt}(U) = \frac{1}{\rho}\sigma_{ij}D_{ij} = \frac{1}{\rho}\sigma_{ij}\dot{\epsilon}_{ij} \qquad (6.64)$$

in which D_{ij} is the rate of deformation tensor. In a second classification, a material is said to be *hypoelastic* if the stress rate is a homogeneous linear function of the rate of deformation. In this case the constitutive equation is written

$$\sigma_{ij}^{\nabla} = K_{ijkm}D_{km} \qquad (6.65)$$

in which the stress rate σ_{ij}^{∇} is defined as

$$\sigma_{ij}^{\nabla} = \frac{d}{dt}(\sigma_{ij}) - \sigma_{iq}V_{qj} - \sigma_{jq}V_{qi} \qquad (6.66)$$

where V_{ij} is the vorticity tensor.

6.10 LINEAR THERMOELASTICITY

If thermal effects are taken into account, the components of the linear strain tensor ϵ_{ij} may be considered to be the sum

$$\epsilon_{ij} = \epsilon_{ij}^{(S)} + \epsilon_{ij}^{(T)} \qquad (6.67)$$

in which $\epsilon_{ij}^{(S)}$ is the contribution from the stress field and $\epsilon_{ij}^{(T)}$ is the contribution from the temperature field. Due to a change from some reference temperature T_0 to the temperature T, the strain components of an elementary volume of an unconstrained isotropic body are given by

$$\epsilon_{ij}^{(T)} = \alpha(T - T_0)\delta_{ij} \qquad (6.68)$$

where α denotes the linear coefficient of thermal expansion. Inserting (*6.68*), together with Hooke's law (*6.22*), into (*6.67*) yields

$$\epsilon_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu}\delta_{ij}\sigma_{kk}\right) + \alpha(T - T_0)\delta_{ij} \qquad (6.69)$$

which is known as the *Duhamel-Neumann* relations. Equation (*6.69*) may be inverted to give the thermoelastic constitutive equations

$$\sigma_{ij} = \lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij} - (3\lambda + 2\mu)\alpha\delta_{ij}(T - T_0) \qquad (6.70)$$

Heat conduction in an isotropic elastic solid is governed by the well-known *Fourier law* of heat conduction,

$$c_i = -kT_{,i} \qquad (6.71)$$

where the scalar k, the thermal conductivity of the body, must be positive to assure a positive rate of entropy production. If now the *specific heat* at constant deformation $c^{(v)}$ is introduced through the equation

$$-c_{i,i} = \rho c^{(v)}\dot{T} \qquad (6.72)$$

and the internal energy is assumed to be a function of the strain components ϵ_{ij} and the temperature T, the energy equation (*5.45*) may be expressed in the form

$$kT_{,ii} = \rho c^{(v)}\dot{T} + (3\lambda + 2\mu)\alpha T_0 \dot{\epsilon}_{ii} \tag{6.73}$$

which is known as the *coupled heat equation.*

The system of equations that formulate the general thermoelastic problem for an isotropic body consists of

(*a*) equations of motion

$$\sigma_{ij,j} + \rho b_i = \ddot{u}_i \quad \text{or} \quad \nabla \cdot \mathbf{\Sigma} + \rho \mathbf{b} = \ddot{\mathbf{u}} \tag{6.74}$$

(*b*) thermoelastic constitutive equations

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu\epsilon_{ij} - (3\lambda + 2\mu)\alpha\delta_{ij}(T - T_0)$$

or

$$\mathbf{\Sigma} = \lambda \mathbf{I}\epsilon + 2\mu \mathbf{E} - (3\lambda + 2\mu)\alpha \mathbf{I}(T - T_0) \tag{6.75}$$

(*c*) strain-displacement relations

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{or} \quad \mathbf{E} = \tfrac{1}{2}(\mathbf{u}\nabla + \nabla\mathbf{u}) \tag{6.76}$$

(*d*) coupled heat equation

$$kT_{,ii} = \rho c^{(v)}\dot{T} + (3\lambda + 2\mu)\alpha T_0 \dot{e}_{kk} \quad \text{or} \quad k\nabla^2 T = \rho c^{(v)}\dot{T} + (3\lambda + 2\mu)\alpha T_0 \dot{\epsilon} \tag{6.77}$$

This system must be solved for the stress, displacement and temperature fields, subject to appropriate initial and boundary conditions. In addition, the compatibility equations must be satisfied.

There is a large collection of problems in which both the inertia and coupling effects may be neglected. For these cases the general thermoelastic problem decomposes into two separate problems which must be solved consecutively, but independently. Thus for the uncoupled, quasi-static, thermoelastic problem the basic equations are the

(*a*) heat conduction equation

$$kT_{,ii} = \rho c^{(v)}\dot{T} \quad \text{or} \quad k\nabla^2 T = \rho c^{(v)}\dot{T} \tag{6.78}$$

(*b*) equilibrium equations

$$\sigma_{ij,j} + \rho b_i = 0 \quad \text{or} \quad \nabla \cdot \mathbf{\Sigma} + \rho \mathbf{b} = 0 \tag{6.79}$$

(*c*) thermoelastic stress-strain equations

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu\epsilon_{ij} - (3\lambda + 2\mu)\alpha\delta_{ij}(T - T_0)$$

or

$$\mathbf{\Sigma} = \lambda \mathbf{I}\epsilon + 2\mu \mathbf{E} - (3\lambda + 2\mu)\alpha \mathbf{I}(T - T_0) \tag{6.80}$$

(*d*) strain-displacement relations

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{or} \quad \mathbf{E} = \tfrac{1}{2}(\nabla\mathbf{u} + \mathbf{u}\nabla) \tag{6.81}$$

Solved Problems

HOOKE'S LAW. STRAIN ENERGY. ISOTROPY (Sec. 6.1-6.3)

6.1. Show that the strain energy density u^* for an isotropic Hookean solid may be expressed in terms of the strain tensor by $u^* = \lambda(\mathrm{tr}\,\mathbf{E})^2/2 + \mu\mathbf{E}:\mathbf{E}$, and in terms of the stress tensor by $u^* = [(1+\nu)\mathbf{\Sigma}:\mathbf{\Sigma} - \nu(\mathrm{tr}\,\mathbf{\Sigma})^2]/2E$.

Inserting (*6.21*) into (*6.13*), $u^* = (\lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij})\epsilon_{ij}/2 = \lambda\epsilon_{ii}\epsilon_{jj}/2 + \mu\epsilon_{ij}\epsilon_{ij}$ which in symbolic notation is $u^* = \lambda(\mathrm{tr}\,\mathbf{E})^2/2 + \mu\mathbf{E}:\mathbf{E}$.

Inserting (*6.24*) into (*6.13*), $u^* = \sigma_{ij}[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}]/2E = [(1+\nu)\sigma_{ij}\sigma_{ij} - \nu\sigma_{ii}\sigma_{jj}]/2E$ which in symbolic notation is $u^* = [(1+\nu)\mathbf{\Sigma}:\mathbf{\Sigma} - \nu(\mathrm{tr}\,\mathbf{\Sigma})^2]/2E$.

6.2. Separating the stress and strain tensors into their spherical and deviator components, express the strain energy density u^* as the sum of a dilatation energy density $u^*_{(S)}$ and distortion energy density $u^*_{(D)}$.

Inserting (*3.98*) and (*2.70*) into (*6.13*),

$$u^* = \tfrac{1}{2}(s_{ij} + \sigma_{kk}\delta_{ij}/3)(e_{ij} + \epsilon_{pp}\delta_{ij}/3) = \tfrac{1}{2}(s_{ij}e_{ij} + \sigma_{ii}e_{jj}/3 + s_{ii}\epsilon_{jj}/3 + \sigma_{ii}\epsilon_{jj}/3)$$

and since $e_{ii} = s_{ii} = 0$ this reduces to $u^* = u^*_{(S)} + u^*_{(D)} = \sigma_{ii}\epsilon_{jj}/6 + s_{ij}e_{ij}/2$.

6.3. Assuming a state of uniform compressive stress $\sigma_{ij} = -p\delta_{ij}$, develop the formulas for the bulk modulus (ratio of pressure to volume change) given in (*6.25*).

With $\sigma_{ij} = -p\delta_{ij}$, (*6.24*) becomes $\epsilon_{ij} = [(1+\nu)(-p\delta_{ij}) + \nu\delta_{ij}(3p)]/E$ and so $\epsilon_{ii} = [-3p(1+\nu) + 9p\nu]/E$. Thus $K = -p/\epsilon_{ii} = E/3(1-2\nu)$. Likewise from (*6.21*), $\sigma_{ii} = (3\lambda + 2\mu)\epsilon_{ii} = -3p$ so that $K = (3\lambda + 2\mu)/3$.

6.4. Express $u^*_{(S)}$ and $u^*_{(D)}$ of Problem 6.2 in terms of the engineering constants K and G and the strain components.

From a result in Problem 6.3, $\sigma_{ii} = 3K\epsilon_{ii}$ and so

$$u^*_{(S)} = \sigma_{ii}\epsilon_{jj}/6 = K\epsilon_{ii}\epsilon_{jj}/2 = K(\mathrm{I}_{\mathrm{E}})^2/2$$

From (*6.21*) and (*2.70*), $\sigma_{ij} = \lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij} = s_{ij} + \sigma_{kk}\delta_{ij}/3$ and since $\sigma_{ii} = (3\lambda + 2\mu)\epsilon_{ii}$ it follows that $s_{ij} = 2\mu(\epsilon_{ij} - \epsilon_{kk}\delta_{ij}/3)$. Thus

$$u^*_{(D)} = 2\mu(\epsilon_{ij} - \epsilon_{kk}\delta_{ij}/3)(\epsilon_{ij} - \epsilon_{pp}\delta_{ij}/3)/2 = \mu(\epsilon_{ij}\epsilon_{ij} - \epsilon_{ii}\epsilon_{jj}/3)$$

Note that the dilatation energy density $u^*_{(S)}$ appears as a function of K only, whereas the distortion energy $u^*_{(D)}$ is in terms of μ (or G), the shear modulus.

6.5. In general, u^* may be expressed in the quadratic form $u^* = C^*_{KM}\epsilon_K\epsilon_M$ in which the C^*_{KM} are not necessarily symmetrical. Show that this equation may be written in the form of (*6.14*) and that $\partial u^*/\partial\epsilon_K = \sigma_K$.

Write the quadratic form as

$$u^* = \tfrac{1}{2}C^*_{KM}\epsilon_K\epsilon_M + \tfrac{1}{2}C^*_{KM}\epsilon_K\epsilon_M = \tfrac{1}{2}C^*_{KM}\epsilon_K\epsilon_M + \tfrac{1}{2}C^*_{PN}\epsilon_N\epsilon_P = \tfrac{1}{2}(C^*_{KM} + C^*_{MK})\epsilon_K\epsilon_M = \tfrac{1}{2}C_{KM}\epsilon_K\epsilon_M$$

where $C_{KM} = C_{MK}$.

Thus the derivative $\partial u^*/\partial\epsilon_R$ is now

$$\partial u^*/\partial\epsilon_R = \tfrac{1}{2}C_{KM}(\epsilon_{K,R}\epsilon_M + \epsilon_K\epsilon_{M,R}) = \tfrac{1}{2}C_{KM}(\delta_{KR}\epsilon_M + \epsilon_K\delta_{MR}) = \tfrac{1}{2}(C_{RM}\epsilon_M + C_{KR}\epsilon_K) = C_{RM}\epsilon_M = \sigma_R$$