# Solved Problems

## HOOKE'S LAW. STRAIN ENERGY. ISOTROPY (Sec. 6.1-6.3)

**6.1.** Show that the strain energy density $u^*$ for an isotropic Hookean solid may be expressed in terms of the strain tensor by $u^* = \lambda(\operatorname{tr}\mathbf{E})^2/2 + \mu\mathbf{E}:\mathbf{E}$, and in terms of the stress tensor by $u^* = [(1+\nu)\mathbf{\Sigma}:\mathbf{\Sigma} - \nu(\operatorname{tr}\mathbf{\Sigma})^2]/2E$.

Inserting (*6.21*) into (*6.13*), $u^* = (\lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij})\epsilon_{ij}/2 = \lambda\epsilon_{ii}\epsilon_{jj}/2 + \mu\epsilon_{ij}\epsilon_{ij}$ which in symbolic notation is $u^* = \lambda(\operatorname{tr}\mathbf{E})^2/2 + \mu\mathbf{E}:\mathbf{E}$.

Inserting (*6.24*) into (*6.13*), $u^* = \sigma_{ij}[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}]/2E = [(1+\nu)\sigma_{ij}\sigma_{ij} - \nu\sigma_{ii}\sigma_{jj}]/2E$ which in symbolic notation is $u^* = [(1+\nu)\mathbf{\Sigma}:\mathbf{\Sigma} - \nu(\operatorname{tr}\mathbf{\Sigma})^2]/2E$.

**6.2.** Separating the stress and strain tensors into their spherical and deviator components, express the strain energy density $u^*$ as the sum of a dilatation energy density $u^*_{(S)}$ and distortion energy density $u^*_{(D)}$.

Inserting (*3.98*) and (*2.70*) into (*6.13*),

$$u^* = \tfrac{1}{2}(s_{ij} + \sigma_{kk}\delta_{ij}/3)(e_{ij} + \epsilon_{pp}\delta_{ij}/3) = \tfrac{1}{2}(s_{ij}e_{ij} + \sigma_{ii}e_{jj}/3 + s_{ii}\epsilon_{jj}/3 + \sigma_{ii}\epsilon_{jj}/3)$$

and since $e_{ii} = s_{ii} = 0$ this reduces to $u^* = u^*_{(S)} + u^*_{(D)} = \sigma_{ii}\epsilon_{jj}/6 + s_{ij}e_{ij}/2$.

**6.3.** Assuming a state of uniform compressive stress $\sigma_{ij} = -p\delta_{ij}$, develop the formulas for the bulk modulus (ratio of pressure to volume change) given in (*6.25*).

With $\sigma_{ij} = -p\delta_{ij}$, (*6.24*) becomes $\epsilon_{ij} = [(1+\nu)(-p\delta_{ij}) + \nu\delta_{ij}(3p)]/E$ and so $\epsilon_{ii} = [-3p(1+\nu) + 9p\nu]/E$. Thus $K = -p/\epsilon_{ii} = E/3(1-2\nu)$. Likewise from (*6.21*), $\sigma_{ii} = (3\lambda + 2\mu)\epsilon_{ii} = -3p$ so that $K = (3\lambda + 2\mu)/3$.

**6.4.** Express $u^*_{(S)}$ and $u^*_{(D)}$ of Problem 6.2 in terms of the engineering constants $K$ and $G$ and the strain components.

From a result in Problem 6.3, $\sigma_{ii} = 3K\epsilon_{ii}$ and so

$$u^*_{(S)} = \sigma_{ii}\epsilon_{jj}/6 = K\epsilon_{ii}\epsilon_{jj}/2 = K(\mathrm{I_E})^2/2$$

From (*6.21*) and (*2.70*), $\sigma_{ij} = \lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij} = s_{ij} + \sigma_{kk}\delta_{ij}/3$ and since $\sigma_{ii} = (3\lambda + 2\mu)\epsilon_{ii}$ it follows that $s_{ij} = 2\mu(\epsilon_{ij} - \epsilon_{kk}\delta_{ij}/3)$. Thus

$$u^*_{(D)} = 2\mu(\epsilon_{ij} - \epsilon_{kk}\delta_{ij}/3)(\epsilon_{ij} - \epsilon_{pp}\delta_{ij}/3)/2 = \mu(\epsilon_{ij}\epsilon_{ij} - \epsilon_{ii}\epsilon_{jj}/3)$$

Note that the dilatation energy density $u^*_{(S)}$ appears as a function of $K$ only, whereas the distortion energy $u^*_{(D)}$ is in terms of $\mu$ (or $G$), the shear modulus.

**6.5.** In general, $u^*$ may be expressed in the quadratic form $u^* = C^*_{KM}\epsilon_K\epsilon_M$ in which the $C^*_{KM}$ are not necessarily symmetrical. Show that this equation may be written in the form of (*6.14*) and that $\partial u^*/\partial\epsilon_K = \sigma_K$.

Write the quadratic form as

$$u^* = \tfrac{1}{2}C^*_{KM}\epsilon_K\epsilon_M + \tfrac{1}{2}C^*_{KM}\epsilon_K\epsilon_M = \tfrac{1}{2}C^*_{KM}\epsilon_K\epsilon_M + \tfrac{1}{2}C^*_{PN}\epsilon_N\epsilon_P = \tfrac{1}{2}(C^*_{KM} + C^*_{MK})\epsilon_K\epsilon_M = \tfrac{1}{2}C_{KM}\epsilon_K\epsilon_M$$

where $C_{KM} = C_{MK}$.

Thus the derivative $\partial u^*/\partial\epsilon_R$ is now

$$\partial u^*/\partial\epsilon_R = \tfrac{1}{2}C_{KM}(\epsilon_{K,R}\epsilon_M + \epsilon_K\epsilon_{M,R}) = \tfrac{1}{2}C_{KM}(\delta_{KR}\epsilon_M + \epsilon_K\delta_{MR}) = \tfrac{1}{2}(C_{RM}\epsilon_M + C_{KR}\epsilon_K) = C_{RM}\epsilon_M = \sigma_R$$

**6.6.** Show that for an orthotropic elastic continuum (three orthogonal planes of elastic symmetry) the elastic coefficient matrix is as given in (*6.19*), page 142.

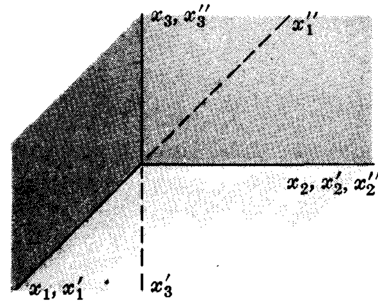



Fig. 6-4

Let the $x_1x_2$ (or equivalently, $x_1'x_2'$) plane be a plane of elastic symmetry (Fig. 6-4). Then $\sigma_K = C_{KM}\epsilon_M$ and also $\sigma_K' = C_{KM}\epsilon_M'$. The transformation matrix between $x_i$ and $x_i'$ is

$$[a_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and from (*2.27*) and (*3.78*), $\sigma_K' = \sigma_K$, $\epsilon_K' = \epsilon_K$ for $K = 1,2,3,6$ whereas $\sigma_K' = -\sigma_K$, $\epsilon_K' = -\epsilon_K$ for $K = 4,5$. Thus, for example, from $\sigma_1' = C_{1M}\epsilon_M'$,

$$\sigma_1' = \sigma_1 = C_{11}\epsilon_1 + C_{12}\epsilon_2 + C_{13}\epsilon_3 - C_{14}\epsilon_4 - C_{15}\epsilon_5 + C_{16}\epsilon_6$$

But from $\sigma_1 = C_{1M}\epsilon_M$,

$$\sigma_1 = C_{11}\epsilon_1 + C_{12}\epsilon_2 + C_{13}\epsilon_3 + C_{14}\epsilon_4 + C_{15}\epsilon_5 + C_{16}\epsilon_6$$

These two expressions for $\sigma_1' = \sigma_1$ are equal only if $C_{14} = C_{15} = 0$. Likewise, from $\sigma_2' = \sigma_2$, $\sigma_3' = \sigma_3$, $\sigma_4' = -\sigma_4$, $\sigma_5' = -\sigma_5$, $\sigma_6' = \sigma_6$ it is found that $C_{24} = C_{25} = C_{34} = C_{35} = C_{64} = C_{65} = C_{41} = C_{42} = C_{43} = C_{51} = C_{52} = C_{53} = C_{56} = 0$.

If $x_2x_3$ (or $x_2''x_3''$) is a second plane of elastic symmetry such that $\sigma_K'' = C_{KM}\epsilon_M''$, the transformation array is

$$[a_{ij}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and now from (*2.27*) and (*3.78*), $\sigma_K'' = \sigma_K$, $\epsilon_K'' = -\epsilon_K$ for $K = 1,2,3,4$ whereas $\sigma_K'' = -\sigma_K$, $\epsilon_K'' = -\epsilon_K$ for $K = 5,6$. Now $C_{16} = C_{26} = C_{36} = C_{45} = C_{54} = C_{61} = C_{62} = C_{63} = 0$ and the elastic coefficient matrix attains the form (*6.19*). The student should verify that elastic symmetry with respect to the (third) $x_1x_3$ plane is identically satisfied by this array.

**6.7.** Give the details of the reduction of the orthotropic elastic matrix (*6.19*) to the isotropic matrix (*6.20*).

For isotropy, elastic properties are the same with respect to all Cartesian coordinate axes. In particular, for the rotated $x_i'$ axes shown in Fig. 6-5, the method of Problem 6.6 results in the matrix (*6.19*) being further simplified by the conditions $C_{11} = C_{22} = C_{33}$, $C_{44} = C_{55} = C_{66}$, and $C_{12} = C_{21} = C_{13} = C_{31} = C_{23} = C_{32}$.

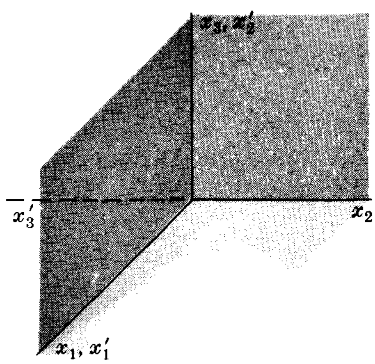



Fig. 6-5

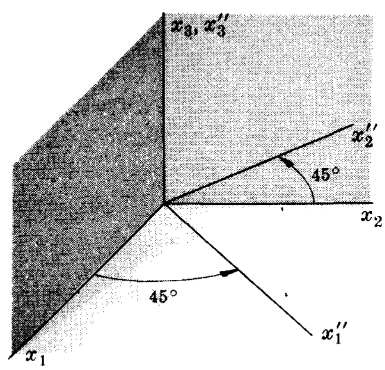



Fig. 6-6

Finally, for the axes $x_i''$ obtained by a 45° rotation about $x_3$ as in Fig. 6-6 above, the transformation matrix is

$$[a_{ij}] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so that $\sigma_6'' = (\sigma_2 - \sigma_1)/2 = (C_{11} - C_{12})(\epsilon_2 - \epsilon_1)/2$ and $\epsilon_6'' = \epsilon_2 - \epsilon_1$. But $\sigma_6'' = C_{44}\epsilon_6''$ and so $2C_{44} = C_{11} - C_{12}$. Thus defining $\mu = C_{44}$ and $\lambda = C_{12}$, *(6.20)* is obtained.

**6.8.** Give the details of the inversion of *(6.21)* to obtain *(6.22)*.

From *(6.21)* with $i = j$, $\sigma_{ii} = (3\lambda + 2\mu)\epsilon_{ii}$ and so $2\mu\epsilon_{ij} = \sigma_{ij} - \lambda\delta_{ij}\sigma_{kk}/(3\lambda + 2\mu)$ or $\epsilon_{ij} = \sigma_{ij}/2\mu - \lambda\delta_{ij}\sigma_{kk}/2\mu(3\lambda + 2\mu)$.

**6.9.** Express the engineering constants $\nu$ and $E$ in terms of the Lamé constants $\lambda$ and $\mu$.

From *(6.25)*, $E/(1-2\nu) = 3\lambda + 2\mu$; and from *(6.26)*, $E/(1+\nu) = 2\mu$. Thus $(3\lambda + 2\mu)(1-2\nu) = 2\mu(1+\nu)$ from which $\nu = \lambda/2(\lambda + \mu)$. Now by *(6.26)*, $E = 2\mu(1+\nu) = \mu(3\lambda + 2\mu)/(\lambda + \mu)$.

**6.10.** Determine the elastic coefficient matrix for a continuum having an axis of elastic symmetry of order $N = 4$. Assume $C_{KM} = C_{MK}$.

Let $x_3$ be the axis of elastic symmetry. A rotation $\theta = 2\pi/4 = \pi/2$ of the axes about $x_3$ produces equivalent elastic directions for $N = 4$. The transformation matrix is

$$[a_{ij}] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and by *(2.27)* and *(3.78)*, $\sigma_1' = \sigma_2$, $\sigma_2' = \sigma_1$, $\sigma_3' = \sigma_3$, $\sigma_4' = -\sigma_5$, $\sigma_5' = \sigma_4$, $\sigma_6' = -\sigma_6$ and $\epsilon_1' = \epsilon_2$, $\epsilon_2' = \epsilon_1$, $\epsilon_3' = \epsilon_3$, $\epsilon_4' = -\epsilon_5$, $\epsilon_5' = \epsilon_4$, $\epsilon_6' = -\epsilon_6$. Thus, for example, from $\sigma_3' = \sigma_3$, $C_{34} = C_{35} = C_{36} = 0$, $C_{31} = C_{32}$. Likewise, from the remaining five stress relations, the elastic matrix becomes

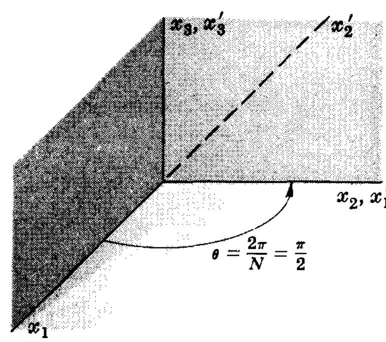



Fig. 6-7

$$[C_{KM}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{11} & C_{13} & 0 & 0 & -C_{16} \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ C_{16} & -C_{16} & 0 & 0 & 0 & C_{66} \end{bmatrix}$$

with seven independent constants.

## ELASTOSTATICS. ELASTODYNAMICS (Sec. 6.4-6.5)

**6.11.** Derive the Navier equations *(6.31)*.

Replacing the strain components in *(6.28)* by the equivalent expressions in terms of displacements yields $\sigma_{ij} = \lambda\delta_{ij}u_{k,k} + \mu(u_{i,j} + u_{j,i})$. Thus $\sigma_{ij,j} = \lambda u_{k,ki} + \mu(u_{i,jj} + u_{j,ij})$. Substituting this into the equilibrium equations *(6.27)* and rearranging terms gives $\mu u_{i,jj} + (\lambda + \mu)u_{j,ji} + \rho b_i = 0$.

**6.12.** Show that if $\nabla^4 F_i = 0$, the displacement $u_i = (\lambda + 2\mu)F_{i,jj}/\mu(\lambda+\mu) - F_{j,ji}/\mu$ is a solution of the Navier equation (*6.31*) for zero body forces.

Differentiating the assumed solution, the terms $\mu u_{i,jj} = (\lambda+2\mu)F_{i,kkjj}/(\lambda+\mu) - F_{k,kijj}$ and $(\lambda+\mu)u_{j,ji} = (\lambda+2\mu)F_{j,kkji}/\mu - (\lambda+\mu)F_{k,kjji}/\mu$ are readily calculated. Inserting these into (*6.31*) gives

$$(\lambda+2\mu)F_{i,kkjj}/(\lambda+\mu) - [\mu - (\lambda+2\mu) + (\lambda+\mu)]F_{j,jkki} = 0$$

provided $F_{i,kkjj} = \nabla^4 F_i = 0$.

**6.13.** If body forces may be neglected, show that (*6.35*) is satisfied by $u_i = \phi_{,i} + \epsilon_{ijk}\psi_{k,j}$ provided $\phi$ and $\psi_i$ each satisfies the familiar three dimensional wave equation.

Substituting the assumed $u_i$ into (*6.35*) yields

$$\mu(\phi_{,ikk} + \epsilon_{ijk}\psi_{k,jqq}) + (\lambda+\mu)(\phi_{,jji} + \epsilon_{jpq}\psi_{q,pji}) = \rho(\ddot{\phi}_{,i} + \epsilon_{ijk}\ddot{\psi}_{k,j})$$

Since $\epsilon_{jpq}\psi_{q,pji} = 0$, this equation may be written

$$((\lambda+2\mu)\phi_{,kk} - \rho\ddot{\phi})_{,i} + \epsilon_{ijk}(\mu\psi_{k,qq} - \rho\ddot{\psi}_k)_{,j} = 0$$

which is satisfied when $\nabla^2\phi = \rho\ddot{\phi}/(\lambda+2\mu)$ and $\nabla^2\psi_k = \rho\ddot{\psi}_k/\mu$.

**6.14.** Writing $c^2\nabla^2\phi = \ddot{\phi}$ where $c^2 = (\lambda+2\mu)/\rho$ for the wave equation derived in Problem 6.13, show that $\phi = \dfrac{g(r+ct) + h(r-ct)}{r}$ is a solution with $g$ and $h$ arbitrary functions of their arguments and $r^2 = x_i x_i$.

Here it is convenient to use the spherical form $\nabla^2 \equiv \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial}{\partial r}\right)$ since $\phi = \phi(r,t)$. Thus $r^2(\partial\phi/\partial r) = r(g'+h') - (g+h)$ where primes denote derivatives with respect to the arguments of $g$ and $h$. Then $\nabla^2\phi = (g''+h'')/r$. Also $\dot{\phi} = (g'c - h'c)/r$ and $\ddot{\phi} = c^2(g''+h'')/r$. Therefore $c^2\nabla^2\phi = \ddot{\phi}$ for the given $\phi$.

**6.15.** Derive the Beltrami-Michell equations (*6.33*) and determine the form they take when body forces are conservative, i.e. when $\rho b_i = \phi_{,i}$.

Substituting (*6.24*) into (*3.103*) yields

$$(1+\nu)(\sigma_{ij,km} + \sigma_{km,ij} - \sigma_{ik,jm} - \sigma_{jm,ik}) = \nu(\delta_{ij}\Theta_{,km} + \delta_{km}\Theta_{,ij} - \delta_{ik}\Theta_{,jm} - \delta_{jm}\Theta_{,ik})$$

where $\Theta = I_\Sigma = \sigma_{ii}$. Only six of the eighty-one equations represented here are independent. Thus setting $m = k$ and using (*6.27*) gives

$$\sigma_{ij,kk} + \Theta_{,ij} + \rho(b_{i,j} + b_{j,i}) = \nu(\delta_{ij}\Theta_{,kk} + \Theta_{,ij})/(1+\nu)$$

from which $\Theta_{,kk} = -(1+\nu)\rho b_{k,k}/(1-\nu)$. Inserting this expression for $\Theta_{,kk}$ into the previous equation leads to (*6.33*).

If $\rho b_i = \phi_{,i}$, then $\rho(b_{i,j} + b_{j,i}) = 2\phi_{,ij}$ and $\rho b_{k,k} = \phi_{,kk} = \nabla^2\phi$ so that (*6.33*) becomes

$$\nabla^2\sigma_{ij} + \Theta_{,ij}/(1+\nu) + 2\phi_{,ij} + \nu\delta_{ij}\nabla^2\phi/(1-\nu) = 0$$

## TWO-DIMENSIONAL ELASTICITY (Sec. 6.6-6.8)

**6.16.** For plane stress parallel to the $x_1x_2$ plane, develop the stress-strain relations in terms of $\lambda$ and $\mu$. Show that these equations correspond to those given as (*6.41*).

Here $\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$ so that (*6.21*) yields $\epsilon_{13} = \epsilon_{23} = 0$ and $\epsilon_{33} = -\lambda(\epsilon_{11}+\epsilon_{22})/(\lambda+2\mu)$. Thus (*6.21*) reduces to $\sigma_{\alpha\beta} = 2\lambda\mu\delta_{\alpha\beta}\epsilon_{\gamma\gamma}/(\lambda+2\mu) + 2\mu\epsilon_{\alpha\beta}$ with $\alpha,\beta,\gamma = 1,2$ from which $\sigma_{\alpha\alpha} = 2\mu(3\lambda+2\mu)\epsilon_{\gamma\gamma}/(\lambda+2\mu)$ so that the equation may be inverted to give

$$\epsilon_{\alpha\beta} = -\lambda\delta_{\alpha\beta}\sigma_{\gamma\gamma}/2\mu(3\lambda+2\mu) + \sigma_{\alpha\beta}/2\mu = -\nu\delta_{\alpha\beta}\sigma_{\gamma\gamma}/E + (1+\nu)\sigma_{\alpha\beta}/E$$

Also,

$$\epsilon_{33} = -\lambda\epsilon_{\gamma\gamma}/(\lambda+2\mu) = -\lambda\sigma_{\gamma\gamma}/2\mu(3\lambda+2\mu) = -\nu\sigma_{\gamma\gamma}/E$$

**6.17.** For plane strain parallel to $x_1x_2$, develop the stress-strain relations in terms of $\nu$ and $E$. Show that these equations correspond to those given as (*6.48*).

Here $u_3 \equiv 0$ so that $\epsilon_{33} = 0$ and (*6.24*) gives $\sigma_{33} = \nu(\sigma_{11} + \sigma_{22}) = \lambda\sigma_{\alpha\alpha}/2(\lambda+\mu)$. Thus (*6.24*) becomes $\epsilon_{\alpha\beta} = (1+\nu)\sigma_{\alpha\beta}/E - \nu(1+\nu)\delta_{\alpha\beta}\sigma_{\gamma\gamma}/E$ from which $\epsilon_{\alpha\alpha} = (1+\nu)(1-2\nu)\sigma_{\alpha\alpha}/E$. Finally, inverting,

$$\sigma_{\alpha\beta} = \nu E\delta_{\alpha\beta}\epsilon_{\gamma\gamma}/(1+\nu)(1-2\nu) + E\epsilon_{\alpha\beta}/(1+\nu) = \lambda\delta_{\alpha\beta}\epsilon_{\gamma\gamma} + 2\mu\epsilon_{\alpha\beta}$$

**6.18.** Develop the Navier equation for plane stress (*6.45*) and show that it is equivalent to the corresponding equation for plane strain (*6.51*) if $\lambda' = 2\lambda\mu/(\lambda+2\mu)$ is substituted for $\lambda$.

Inverting (*6.41*) and using (*6.42*) leads to $\sigma_{\alpha\beta} = E(u_{\alpha,\beta} + u_{\beta,\alpha})/2(1+\nu) + 2\nu E\delta_{\alpha\beta}u_{\gamma,\gamma}/2(1-\nu^2)$. Differentiating with respect to $x_\beta$ and substituting into (*6.40*) gives

$$Eu_{\alpha,\beta\beta}/2(1+\nu) + Eu_{\beta,\beta\alpha}/2(1-\nu) + \rho b_\alpha = \mu\nabla^2 u_\alpha + \mu(3\lambda+2\mu)u_{\beta,\beta\alpha}/(\lambda+2\mu) + \rho b_\alpha = 0$$

Thus since $\mu(3\lambda+2\mu)/(\lambda+2\mu) = (2\lambda\mu/(\lambda+2\mu)+\mu) = (\lambda'+\mu)$, (*6.45*) and (*6.51*) have the same form for the given substitution.

**6.19.** Determine the necessary relationship between the constants $A$ and $B$ if $\phi = Ax_1^2x_2^3 + Bx_2^5$ is to serve as an Airy stress function.

By (*6.56*), $\phi$ must be biharmonic or $\phi_{,1111} + 2\phi_{,1122} + \phi_{,2222} = 0 + 24Ax_2 + 120Bx_2 = 0$, which is satisfied when $A = -5B$.

**6.20.** Show that $\phi = \dfrac{3F}{4c}\left[x_1x_2 - \dfrac{x_1x_2^3}{3c^2}\right] + \dfrac{P}{4c}x_2^2$ is suitable for use as an Airy stress function and determine the stress components in the region $x_1 > 0$, $-c < x_2 < c$.

Since $\nabla^4\phi$ is identically zero, $\phi$ is a valid stress function. The stress components as given by (*6.55*) are $\sigma_{11} = \phi_{,22} = -3Fx_1x_2/2c^3 + P/2c$, $\sigma_{12} = -\phi_{,12} = -3F(c^2 - x_2^2)/4c^3$, $\sigma_{22} = \phi_{,11} = 0$. These stresses are those of a cantilever beam subjected to a transverse end load $F$ and an axial pull $P$ (Fig. 6-8).

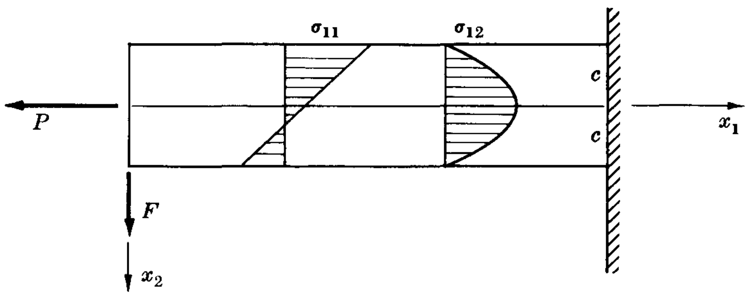



**Fig. 6-8**

**6.21.** In Problem 2.36 it was shown that the equilibrium equations were satisfied in the absence of body forces by $\sigma_{ij} = \epsilon_{ipq}\epsilon_{jmn}\phi_{qn,pm}$. Show that Airy's stress function is represented by the case $\phi_{33} = \phi(x_1, x_2)$ with $\phi_{11} = \phi_{22} = \phi_{12} = \phi_{13} = \phi_{23} \equiv 0$.

Since $\phi_{33}$ is the only non-vanishing component, $\sigma_{ij} = \epsilon_{ipq}\epsilon_{jmn}\phi_{qn,pm}$ becomes $\sigma_{ij} = \epsilon_{ip3}\epsilon_{jm3}\phi_{33,pm}$ which may be written $\sigma_{\alpha\beta} = \epsilon_{\alpha\gamma3}\epsilon_{\beta\zeta3}\phi_{33,\gamma\zeta}$. Thus since $\phi_{33} = \phi$, $\sigma_{\alpha\beta} = (\delta_{\alpha\beta}\delta_{\gamma\zeta} - \delta_{\alpha\zeta}\delta_{\gamma\beta})\phi_{,\gamma\zeta} = \delta_{\alpha\beta}\phi_{,\gamma\gamma} - \phi_{,\alpha\beta}$. The stress components are therefore $\sigma_{11} = \phi_{,11} + \phi_{,22} - \phi_{,11} = \phi_{,22}$, $\sigma_{12} = -\phi_{,12}$, $\sigma_{22} = \phi_{,11} + \phi_{,22} - \phi_{,22} = \phi_{,11}$.

**6.22.** In polar coordinates $(r, \theta)$ the Airy stress function $\Phi = B\theta$ is used in the solution of a disk of radius $a$ subjected to a central moment $M$. Determine the stress components and the value of the constant $B$.

From (6.60) and (6.61), $\sigma_{(rr)} = \sigma_{(\theta\theta)} = 0$. From (6.62) $\sigma_{(r\theta)} = B/r^2$. Equilibrium of moments about the center of the disk requires $M = \int_0^{2\pi} \sigma_{(r\theta)} a^2\, d\theta = \int_0^{2\pi} B\, d\theta = 2\pi B$. Thus $B = M/2\pi$.

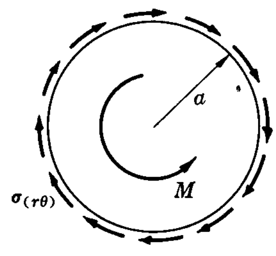



Fig. 6-9

## LINEAR THERMOELASTICITY (Sec. 6.10)

**6.23.** Carry out the inversion of (6.69) to obtain the thermoelastic constitutive equations (6.70).

From (6.69) with $i = j$, $\sigma_{ii} = (3\lambda + 2\mu)(\epsilon_{ii} - 3\alpha(T - T_0))$. Solving (6.69) for $\sigma_{ij}$ gives

$$\begin{aligned}\sigma_{ij} &= 2\mu\epsilon_{ij} + \lambda\delta_{ij}\sigma_{kk}/(3\lambda + 2\mu) - 2\mu\alpha\delta_{ij}(T - T_0)\\ &= 2\mu\epsilon_{ij} + \lambda\delta_{ij}(\epsilon_{kk} - 3\alpha(T - T_0)) - 2\mu\alpha\delta_{ij}(T - T_0)\\ &= 2\mu\epsilon_{ij} + \lambda\delta_{ij}\epsilon_{kk} - (3\lambda + 2\mu)\alpha\delta_{ij}(T - T_0)\end{aligned}$$

**6.24.** Develop the thermoelastic energy equation (6.73) by use of the free energy $f = u - Ts$.

Assuming the free energy to be a function of the strains and temperature, $f = f(\epsilon_{ij}, T)$ and substituting into (5.41) $\rho\dot{u} = \sigma_{ij}\dot{\epsilon} + \rho T\dot{s}$ where dots indicate time derivatives, the result is $(\sigma_{ij} - \rho\partial f/\partial\epsilon_{ij})\dot{\epsilon}_{ij} - \rho(s + \partial f/\partial T)\dot{T} = 0$. Since the terms in parentheses are independent of strain and temperature rates, it follows that $\sigma_{ij} = \rho\partial f/\partial\epsilon_{ij}$ and $s = -\partial f/\partial T$. From (5.38) for a reversible isothermal process, $-c_{i,i} = \rho T\dot{s} = \rho T\left(\frac{\partial s}{\partial \epsilon_{ij}}\dot{\epsilon}_{ij} + \frac{\partial s}{\partial T}\dot{T}\right)$. At constant deformation, $\dot{\epsilon}_{ij} = 0$ and comparing this equation with (6.72) gives $c^{(v)} = T(\partial s/\partial T)$ or from above, since $\partial s/\partial T = -\partial^2 f/\partial T^2$, $c^{(v)} = -\partial^2 f/\partial T^2$. Also, from above, $\rho(\partial^2 f/\partial\epsilon_{ij}\partial T) = \partial\sigma_{ij}/\partial T$ and so combining (5.38) with (6.71), $-c_{i,i} = kT_{,ii} = \rho T\left(\frac{\partial\sigma_{ij}}{\partial T}\dot{\epsilon}_{ij} + \frac{c^{(v)}}{T}\dot{T}\right)$. Finally from (6.70), $\partial\sigma_{ij}/\partial T = (3\lambda + 2\mu)\alpha\delta_{ij}T_0$ so that $kT_{,ii} = \rho c^{(v)}\dot{T} + (3\lambda + 2\mu)\alpha T_0\dot{\epsilon}_{ii}$ which is (6.73).

**6.25.** Use (6.13) and (6.70) to develop the strain energy density for a thermoelastic solid.

Substituting (6.70) directly into (6.13),

$$\begin{aligned}u^* &= \lambda\delta_{ij}\epsilon_{kk}\epsilon_{ij}/2 + \mu\epsilon_{ij}\epsilon_{ij} - (3\lambda + 2\mu)\alpha\delta_{ij}(T - T_0)\epsilon_{ij}/2\\ &= \lambda\epsilon_{ii}\epsilon_{jj}/2 + \mu\epsilon_{ij}\epsilon_{ij} - (3\lambda + 2\mu)\alpha(T - T_0)\epsilon_{ii}/2\end{aligned}$$

## MISCELLANEOUS PROBLEMS

**6.26.** Show that the distortion energy density $u^*_{(D)}$ may be expressed in terms of principal stress values by the equation $u^*_{(D)} = [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]/12G$.

From Problem 6.2, $u^*_{(D)} = s_{ij}e_{ij}/2 = s_{ij}s_{ij}/4G$ which in terms of stress components becomes

$$u^*_{(D)} = (\sigma_{ij} - \delta_{ij}\sigma_{kk}/3)(\sigma_{ij} - \delta_{ij}\sigma_{pp}/3)/4G = (\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj}/3)/4G$$

In terms of principal stresses this is

$$\begin{aligned}u^*_{(D)} &= [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - (\sigma_1 + \sigma_2 + \sigma_3)(\sigma_1 + \sigma_2 + \sigma_3)/3]/4G\\ &= [2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1)/3]/4G\\ &= [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]/12G\end{aligned}$$

**6.27.** Use the results of Problem 6.1 to show that for an elastic material $\partial u^*/\partial\epsilon_{ij} = \sigma_{ij}$ and $\partial u^*/\partial\sigma_{ij} = \epsilon_{ij}$.

From Problem 6.1, $u^* = \lambda\epsilon_{ii}\epsilon_{jj}/2 + \mu\epsilon_{ij}\epsilon_{ij}$ and so

$$\begin{aligned}\partial u^*/\partial\epsilon_{pq} &= \lambda/2[\epsilon_{ii}(\partial\epsilon_{jj}/\partial\epsilon_{pq}) + \epsilon_{jj}(\partial\epsilon_{ii}/\partial\epsilon_{pq})] + 2\mu\epsilon_{ij}(\partial\epsilon_{ij}/\partial\epsilon_{pq})\\ &= \lambda/2[\epsilon_{ii}\delta_{jp}\delta_{jq} + \epsilon_{jj}\delta_{ip}\delta_{iq}] + 2\mu\epsilon_{ij}\delta_{ip}\delta_{jq} = \lambda/2[\epsilon_{ii}\delta_{pq} + \epsilon_{jj}\delta_{pq}] + 2\mu\epsilon_{pq}\\ &= \lambda\epsilon_{ii}\delta_{pq} + 2\mu\epsilon_{pq} = \sigma_{pq}\end{aligned}$$

Likewise from Problem 6.1, $u^* = [(1+\nu)\sigma_{ij}\sigma_{ij} - \nu\sigma_{ii}\sigma_{jj}]/2E$ and so

$$\partial u^*/\partial\sigma_{pq} = [2(1+\nu)\sigma_{ij}\delta_{ip}\delta_{jq} - \nu(\sigma_{ii}\delta_{pq} + \sigma_{jj}\delta_{pq}]/2E = [(1+\nu)\sigma_{pq} - \nu\delta_{pq}\sigma_{ii}]/E = \epsilon_{pq}$$

**6.28.** Express the strain energy density $u^*$ as a function of the strain invariants.

From Problem 6.1, $u^* = \lambda\epsilon_{ii}\epsilon_{jj}/2 + \mu\epsilon_{ij}\epsilon_{ij}$; and since by comparison with *(3.91)*, $\text{I}_E = \epsilon_{ii}$ and $\text{II}_E = (\epsilon_{ii}\epsilon_{jj} - \epsilon_{ij}\epsilon_{ij})/2$, it follows that

$$u^* = \lambda(\text{I}_E)^2/2 + \mu(-2\text{II}_E + (\text{I}_E)^2) = (\lambda/2+\mu)(\text{I}_E)^2 - 2\mu\text{II}_E$$

**6.29.** When a circular shaft of length $L$ and radius $a$ is subjected to end couples as shown in Fig. 6-10, the nonzero stress components are $\sigma_{13} = -G\alpha x_2$, $\sigma_{23} = G\alpha x_1$ where $\alpha$ is the angle of twist per unit length. Determine expressions for the strain energy density and the total strain energy in the shaft.

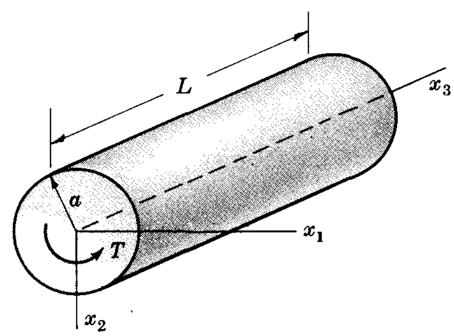


Fig. 6-10

From Problem 6.1, $u^* = [(1+\nu)\mathbf{\Sigma}:\mathbf{\Sigma} - \nu(\text{tr}\,\mathbf{\Sigma})^2]/2E$. Here $\text{tr}\,\mathbf{\Sigma} \equiv 0$ and $\mathbf{\Sigma}:\mathbf{\Sigma} = 2G^2\alpha^2r^2$ where $r^2 = x_1^2 + x_2^2$. Thus $u^* = G\alpha^2r^2/2$. The total strain energy is given by

$$U = \int_V u^*\,dV = \frac{G\alpha^2}{2}\int_0^a\int_0^{2\pi}\int_0^L r^3\,dr\,d\theta\,dx_3 = G\alpha^2a^4\pi L/4$$

Note that since $T = \int_0^{2\pi}\int_0^a G\alpha(x_1^2 + x_2^2)r\,dr\,d\theta = G\alpha a^4\pi/2$, $U = T\alpha L/2$, the external work.

**6.30.** Show that for a continuum having an axis of elastic symmetry of order $N = 2$, the elastic properties (Hooke's law and strain energy density) are of the same form as a continuum having one plane of elastic symmetry.

Here a rotation of axes $\theta = 2\pi/N = 2\pi/2 = \pi$ produces equivalent elastic directions. But this is precisely the same situation as the reflection about a plane of elastic symmetry.

**6.31.** Show that *(6.19)* with $C_{11} = C_{22} = C_{33}$, $C_{44} = C_{55} = C_{66}$ and $C_{12} = C_{13} = C_{23}$ may be reduced to *(6.20)* by an arbitrary rotation $\theta$ of axes about $x_3$ (Fig. 6-11).

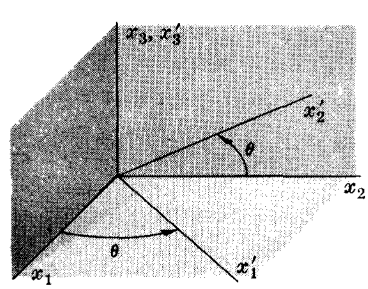


Fig. 6-11

The transformation between $x_i$ and $x_i'$ axes is

$$a_{ij} = \begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$$

and from *(2.27)*,

$$\sigma'_{12} = (-\sin\theta\cos\theta)\sigma_{11} + (\cos^2\theta - \sin^2\theta)\sigma_{12} + (\sin\theta\cos\theta)\sigma_{22}$$

or in single index notation,

$$\sigma'_6 = (-\sin\theta\cos\theta)\sigma_1 + (\cos^2\theta - \sin^2\theta)\sigma_6 + (\sin\theta\cos\theta)\sigma_2$$

Likewise from *(3.78)* and *(6.4)*,

$$\epsilon'_6 = (-2\sin\theta\cos\theta)\epsilon_1 + (\cos^2\theta - \sin^2\theta)\epsilon_6 + (2\sin\theta\cos\theta)\epsilon_2$$

But $\sigma'_6 = C_{44}\epsilon'_6$ for an isotropic body and so here $\sigma_2 - \sigma_1 = 2C_{44}(\epsilon_2 - \epsilon_1)$. Finally from *(6.19)* with the given conditions, $\sigma_1 = C_{11}\epsilon_1 + C_{12}(\epsilon_2 + \epsilon_1)$ and $\sigma_2 = C_{11}\epsilon_2 + C_{12}(\epsilon_1 + \epsilon_2)$ and so $\sigma_2 - \sigma_1 = (C_{11} - C_{12})(\epsilon_2 - \epsilon_1)$. Therefore $(C_{11} - C_{12}) = 2C_{44}$ and with $C_{44} = \mu$, $C_{12} = \lambda$, $C_{11} = \lambda + 2\mu$ as given in *(6.20)*.

**6.32.** For an elastic body in equilibrium under body forces $b_i$ and surface forces $t_i^{(\hat{n})}$, show that the total strain energy is equal to one-half the work done by the external forces acting through their displacements $u_i$.

It is required to show that $\int_V \rho b_i u_i\,dV + \int_S t_i^{(\hat{n})} u_i\,dS = 2\int_V u^*\,dV$. Consider first the surface integral with $t_i^{(\hat{n})} = \sigma_{ji}u_j$ and convert by Gauss' theorem. Thus

$$\int_S \sigma_{ij}u_i n_j\,dS = \int_V (\sigma_{ij}u_i)_{,j}\,dV = \int_V (\sigma_{ij,j}u_i + \sigma_{ij}u_{i,j})\,dV$$

But $\sigma_{ij}u_{i,j} = \sigma_{ij}(\epsilon_{ij} + \omega_{ij}) = \sigma_{ij}\epsilon_{ij}$, and from equilibrium $\sigma_{ij,j} = -\rho b_i$. Thus

$$\int_S t_i^{(\hat{n})} u_i\,dS = -\int_V \rho b_i u_i\,dV + 2\int \sigma_{ij}\epsilon_{ij}/2\,dV$$

and the theorem is proved.

**6.33.** Use the result of Problem 6.32 to establish uniqueness of the elastostatic solution of a linear elastic body by assuming two solutions $\sigma_{ij}^{(1)}, u_i^{(1)}$ and $\sigma_{ij}^{(2)}, u_i^{(2)}$.

For linear elasticity superposition holds, so $\sigma_{ij} = \sigma_{ij}^{(1)} - \sigma_{ij}^{(2)}$, $u_i = u_i^{(1)} - u_i^{(2)}$ would also be a solution for which $b_i = 0$. Thus for this "difference" solution $\int_S t_i^{(\hat{n})} u_i\,dS = 2\int_V u^*\,dV$ from Problem 6.32. Since the two assumed solutions satisfy boundary conditions, the left hand integral is zero here since $t_i^{(\hat{n})} = t_i^{(1)} - t_i^{(2)}$ on the boundary for equation *(6.32)* and $u_i = u_i^{(1)} - u_i^{(2)}$ on the boundary for equation *(6.30)*. Thus $\int_V u^*\,dV = 0$ and since $u^*$ is positive definite this occurs only if $\epsilon_{ij} = \epsilon_{ij}^{(1)} - \epsilon_{ij}^{(2)} \equiv 0$, or $\epsilon_{ij}^{(1)} = \epsilon_{ij}^{(2)}$. If the strains are equal for the two assumed solutions, the stresses are also equal by Hooke's law and the displacements are equal to within a rigid body displacement. Thus uniqueness is established.

**6.34.** The Navier equations *(6.31)* may be put in the form $\mu u_{i,jj} + \frac{\mu}{1-2\nu}u_{j,ji} + \rho b_i = 0$ which for the incompressible case $(\nu = \frac{1}{2})$ are clearly indeterminate. Use the equilibrium equations for this situation to show that $\mu u_{i,jj} + \Theta_{,i}/3 + \rho b_i = 0$.

From equation *(6.24)*, $\epsilon_{ii} = (1 - 2\nu)\sigma_{ii}/E$; and for $\nu = \frac{1}{2}$, $\epsilon_{ii} \equiv u_{i,i} = 0$. Thus from *(6.24)*,

$$2\epsilon_{ij,j} = u_{i,jj} + u_{j,ij} = 2(1+\nu)\sigma_{ij,j}/E - 2\nu\delta_{ij}\sigma_{kk,j}/E$$

But $u_{j,ji} = 0$ and $E = 3G$ when $\nu = \frac{1}{2}$, so that $u_{i,jj} = -\rho b_i/G - \sigma_{kk,i}/3G$ or $\mu\nabla^2 u_i + \Theta_{,i}/3 + \rho b_i = 0$.

## Supplementary Problems

**6.35.** Prove that the principal axes of the stress and strain tensors coincide for a homogeneous isotropic elastic body.

**6.36.** Develop the expression for the strain energy density $u^*$ for an orthotropic elastic medium. Use equations *(6.14)* and *(6.19)*.

*Ans.* $u^* = (C_{11}\epsilon_1 + 2C_{12}\epsilon_2 + 2C_{13}\epsilon_5)\epsilon_1/2 + (C_{22}\epsilon_2 + 2C_{23}\epsilon_4)\epsilon_2/2 + C_{33}\epsilon_3^2 + C_{44}\epsilon_4^2 + C_{55}\epsilon_5^2 + C_{66}\epsilon_6^2$.

**6.37.** Determine the form of the strain energy density for the case of (*a*) plane stress, (*b*) plane strain.
*Ans.* (*a*) $u^* = [\sigma_{11}^2 + \sigma_{22}^2 - 2\nu\sigma_{11}\sigma_{22} + 2(1+\nu)\sigma_{12}^2]/2E$
(*b*) $u^* = (\mu + \lambda/2)(\epsilon_{11}^2 + \epsilon_{22}^2) + \lambda\epsilon_{11}\epsilon_{22} + 2\mu\epsilon_{12}^2$

**6.38.** Determine the value of $c$ for which $u_1 = A\sin\frac{2\pi}{l}(x_1 \pm ct)$, $u_2 = u_3 = 0$ is a solution of equation (*6.35*) when body forces are zero. *Ans.* $c = \sqrt{(\lambda + 2\mu)/\rho}$

**6.39.** Show that the distortion energy density $u^*_{(D)} = (\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj}/3)/4G$ and the dilatation energy density $u^*_{(S)} = \sigma_{ii}\sigma_{jj}/18K$.

**6.40.** Show that $1/(1+\nu) = 2(\lambda + \mu)/(3\lambda + 2\mu)$ and $\nu/(1-\nu) = \lambda/(\lambda + 2\mu)$.

**6.41.** For plane strain parallel to $x_1x_2$, show that $b_3 \equiv 0$ and that $b_1$ and $b_2$ are functions of $x_1$ and $x_2$ only.

**6.42.** Use the transformation laws for stress and strain to show that the elastic constants $C_{ijkm}$ are the components of a fourth order Cartesian tensor so that $C'_{ijkm} = a_{ip}a_{jq}a_{kr}a_{ms}C_{pqrs}$.

**6.43.** Show that the Airy stress function $\phi = 2x_1^4 + 12x_1^2x_2^2 - 6x_2^4$ satisfies the biharmonic equation $\nabla^4\phi = 0$ and determine the stress components assuming plane strain.

$$Ans. \quad \sigma_{ij} = 24\begin{pmatrix} x_1^2 - 3x_2^2 & -2x_1x_2 & 0 \\ -2x_1x_2 & x_1^2 + x_2^2 & 0 \\ 0 & 0 & 2\nu(x_1^2 - x_2^2) \end{pmatrix}$$

**6.44.** Determine the strains associated with the stresses of Problem 6.43 and show that the compatibility equation (*6.44*) is satisfied.

$$Ans. \quad \epsilon_{ij} = 24\left(\frac{1+\nu}{E}\right)\begin{pmatrix} x_1^2 - 3x_2^2 - 2\nu(x_1^2 - x_2^2) & -2x_1x_2 & 0 \\ -2x_1x_2 & x_1^2 + x_2^2 - 2\nu(x_1^2 - x_2^2) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

**6.45.** For an elastic body having an axis of elastic symmetry of order $N = 6$, show that $C_{22} = C_{11}$, $C_{55} = C_{44}$, $C_{66} = 2(C_{11} - C_{12})$ and that $C_{13}$ and $C_{33}$ are the only remaining nonzero coefficients.

**6.46.** Show that for an elastic continuum with conservative body forces such that $\rho b_\alpha = \nabla\psi = \psi_{,\alpha}$, the compatibility condition (*6.44*) may be written $\nabla^2\sigma_{\alpha\alpha} = \nabla^2\psi/(1-\nu)$ for plane strain, or $\nabla^2\sigma_{\alpha\alpha} = (1+\nu)\nabla^2\psi$ for plane stress.

**6.47.** If $\nabla^4 F_i = 0$, show that $u_i = 2(1-\nu)\nabla^2 F_i/G - F_{j,ji}/G$ is a solution of the Navier equation (*6.31*) when $b_i \equiv 0$ (see Problem 6.12). If $\mathbf{F} = B(x_2\hat{\mathbf{e}}_1 - x_1\hat{\mathbf{e}}_2)/r$ where $r^2 = x_ix_i$, determine the stress components.
*Ans.* $\sigma_{11} = -\sigma_{22} = 6QGx_1x_2/r^5$, $\sigma_{33} = 0$, $\sigma_{12} = 3QG(x_2^2 - x_1^2)/r^5$, $\sigma_{13} = -\sigma_{23} = 3QGx_2x_3/r^5$, where $Q = 4B(1-\nu)/G$.

**6.48.** In polar coordinates an Airy stress function is given by $\Phi = Cr^2(\cos 2\theta - \cos 2\alpha)$ where $C$ and $\alpha$ are constants. Determine $C$ if $\sigma_{\theta\theta} = 0$, $\sigma_{r\theta} = \tau$ when $\theta = \alpha$, and $\sigma_{\theta\theta} = 0$, $\sigma_{r\theta} = -\tau$ when $\theta = -\alpha$. *Ans.* $C = \tau/(2\sin 2\alpha)$

**6.49.** Show that in plane strain thermoelastic problems $\sigma_{33} = \nu(\sigma_{11} + \sigma_{22}) - \alpha E(T - T_0)$ and that $\sigma_{\alpha\beta} = \lambda\delta_{\alpha\beta}\epsilon_{\alpha\alpha} + 2\mu\epsilon_{\alpha\beta} - \delta_{\alpha\beta}(3\lambda + 2\mu)\alpha(T - T_0)$. In plane stress thermoelasticity show that

$$\epsilon_{33} = -\nu(\sigma_{11} + \sigma_{22})/E + \alpha(T - T_0) \quad \text{and} \quad \epsilon_{\alpha\beta} = (1+\nu)\sigma_{\alpha\beta}/E - \nu\delta_{\alpha\beta}\sigma_{\alpha\alpha}/E + \delta_{\alpha\beta}(T - T_0)\alpha$$

**6.50.** In terms of the Airy stress function $\phi = \phi(x_1, x_2)$, show that for plane strain thermoelasticity the compatibility equation (*6.44*) may be expressed as $\nabla^4\phi = -\alpha E\nabla^2(T - T_0)/(1-\nu)$ and that for plane stress as $\nabla^4\phi = -\alpha E\nabla^2(T - T_0)$.

# Chapter 7

## Fluids

### 7.1 FLUID PRESSURE. VISCOUS STRESS TENSOR. BAROTROPIC FLOW

In any fluid at rest the stress vector $t_i^{(\hat{n})}$ on an arbitrary surface element is collinear with the normal $\hat{\mathbf{n}}$ of the surface and equal in magnitude for every direction at a given point. Thus

$$t_i^{(\hat{n})} = \sigma_{ij} n_j = -p_0 n_i \quad \text{or} \quad \mathbf{t}^{(\hat{n})} = \mathbf{\Sigma} \cdot \hat{\mathbf{n}} = -p_0 \hat{\mathbf{n}} \tag{7.1}$$

in which $p_0$ is the stress magnitude, or *hydrostatic pressure.* The negative sign indicates a compressive stress for a positive value of the pressure. Here every direction is a principal direction, and from (*7.1*)

$$\sigma_{ij} = -p_0 \delta_{ij} \quad \text{or} \quad \mathbf{\Sigma} = -p_0 \mathbf{I} \tag{7.2}$$

which represents a spherical state of stress often referred to as hydrostatic pressure. From (*7.2*), the shear stress components are observed to be zero in a fluid at rest.

For a fluid in motion, the shear stress components are usually not zero, and it is customary in this case to resolve the stress tensor according to the equation

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \quad \text{or} \quad \mathbf{\Sigma} = -p\mathbf{I} + \mathbf{T} \tag{7.3}$$

where $\tau_{ij}$ is called the *viscous stress tensor* and $p$ is the *pressure.*

All real fluids are both compressible and viscous. However, these characteristics vary widely in different fluids so that it is often possible to neglect their effects in certain situations without significant loss of accuracy in calculations based upon such assumptions. Accordingly, an *inviscid,* or so-called *perfect fluid* is one for which $\tau_{ij}$ is taken identically zero even when motion is present. *Viscous fluids* on the other hand are those for which $\tau_{ij}$ must be considered. For a compressible fluid, the pressure $p$ is essentially the same as the pressure associated with classical thermodynamics. From (*7.3*), the mean normal stress is given by

$$\tfrac{1}{3}\sigma_{ii} = -p + \tfrac{1}{3}\tau_{ii} \quad \text{or} \quad \tfrac{1}{3}\Theta = -p + \tfrac{1}{3}\Gamma \tag{7.4}$$

For a fluid at rest, $\tau_{ij}$ vanishes and $p$ reduces to $p_0$ which in this case is equal to the negative of the mean normal stress. For an incompressible fluid, the thermodynamic pressure is not defined separately from the mechanical conditions so that $p$ must be considered as an independent mechanical variable in such fluids.

In a compressible fluid, the pressure $p$, the density $\rho$ and the absolute temperature $T$ are related through a kinetic equation of state having the form

$$p = p(\rho, T) \tag{7.5}$$

An example of such an equation of state is the well-known ideal gas law

$$p = \rho R T \tag{7.6}$$

where $R$ is the gas constant. If the changes of state of a fluid obey an equation of state that does not contain the temperature, i.e. $p = p(\rho)$, such changes are termed *barotropic*. An isothermal process for a perfect gas is an example of a special case which obeys the barotropic assumption.

## 7.2 CONSTITUTIVE EQUATIONS. STOKESIAN FLUIDS. NEWTONIAN FLUIDS

The viscous stress components of the stress tensor for a fluid are associated with the dissipation of energy. In developing constitutive relations for fluids, it is generally assumed that the viscous stress tensor $\tau_{ij}$ is a function of the rate of deformation tensor $D_{ij}$. If the functional relationship is a nonlinear one, as expressed symbolically by

$$\tau_{ij} = f_{ij}(D_{pq}) \quad \text{or} \quad \mathbf{\Gamma} = \mathbf{f}(\mathbf{D}) \tag{7.7}$$

the fluid is called a *Stokesian fluid.* When the function is a linear one of the form

$$\tau_{ij} = K_{ijpq}D_{pq} \quad \text{or} \quad \mathbf{\Gamma} = \tilde{\mathbf{K}}:\mathbf{D} \tag{7.8}$$

where the constants $K_{ijpq}$ are called *viscosity coefficients*, the fluid is known as a *Newtonian fluid.* Some authors classify fluids simply as *Newtonian* and *non-Newtonian.*

Following a procedure very much the same as that carried out for the generalized Hooke's law of an elastic media in Chapter 6, the constitutive equations for an isotropic homogeneous Newtonian fluid may be determined from (*7.7*) and (*7.3*). The final form is

$$\sigma_{ij} = -p\delta_{ij} + \lambda^*\delta_{ij}D_{kk} + 2\mu^*D_{ij} \quad \text{or} \quad \mathbf{\Sigma} = -p\mathbf{I} + \lambda^*\mathbf{I}(\text{tr}\,\mathbf{D}) + 2\mu^*\mathbf{D} \tag{7.9}$$

where $\lambda^*$ and $\mu^*$ are viscosity coefficients of the fluid. From (*7.9*), the mean normal stress is given by

$$\tfrac{1}{3}\sigma_{ii} = -p + \tfrac{1}{3}(3\lambda^* + 2\mu^*)D_{ii} = -p + \kappa^*D_{ii}$$

or

$$\tfrac{1}{3}(\text{tr}\,\mathbf{\Sigma}) = -p + \tfrac{1}{3}(3\lambda^* + 2\mu^*)(\text{tr}\,\mathbf{D}) = -p + \kappa^*(\text{tr}\,\mathbf{D}) \tag{7.10}$$

where $\kappa^* = \frac{1}{3}(3\lambda^* + 2\mu^*)$ is called the *coefficient of bulk viscosity.* The condition that

$$\kappa^* = \lambda^* + \tfrac{2}{3}\mu^* = 0 \tag{7.11}$$

is known as *Stokes' condition*, and guarantees that the pressure $p$ is defined as the average of the normal stresses for a compressible fluid at rest. In this way the thermodynamic pressure is defined in terms of the mechanical stresses.

In terms of the deviator components $s_{ij} = \sigma_{ij} - \delta_{ij}\sigma_{kk}/3$ and $D'_{ij} = D_{ij} - \delta_{ij}D_{kk}/3$, equation (*7.9*) above may be rewritten in the form

$$s_{ij} + \tfrac{1}{3}\delta_{ij}\sigma_{kk} = -p\delta_{ij} + \delta_{ij}(\lambda^* + \tfrac{2}{3}\mu^*)D_{ii} + 2\mu^*D'_{ij}$$

or

$$\mathbf{S} + \tfrac{1}{3}\mathbf{I}(\text{tr}\,\mathbf{\Sigma}) = -p\mathbf{I} + \mathbf{I}(\lambda^* + \tfrac{2}{3}\mu^*)(\text{tr}\,\mathbf{D}) + 2\mu^*\mathbf{D}' \tag{7.12}$$

Therefore in view of the relationship (*7.10*), equation (*7.12*) may be expressed by the pair of equations

$$s_{ij} = 2\mu^*D'_{ij} \quad \text{or} \quad \mathbf{S} = 2\mu^*\mathbf{D}' \tag{7.13}$$

$$\sigma_{ii} = -3p + 3\kappa^*D_{ii} \quad \text{or} \quad \text{tr}\,\mathbf{\Sigma} = -3p + 3\kappa^*(\text{tr}\,\mathbf{D}) \tag{7.14}$$

the first of which relates the shear effects in the fluid and the second gives the volumetric relationship.

## 7.3 BASIC EQUATIONS FOR NEWTONIAN FLUIDS. NAVIER-STOKES-DUHEM EQUATIONS

In Eulerian form, the basic equations required to formulate the problem of motion for a Newtonian fluid are

(*a*) the continuity equation (*5.3*),

$$\dot{\rho} + \rho v_{i,i} = 0 \quad \text{or} \quad \dot{\rho} + \rho(\nabla_{\mathbf{x}} \cdot \mathbf{v}) = 0 \tag{7.15}$$

(*b*) the equations of motion (*5.16*),

$$\sigma_{ij,j} + \rho b_i = \rho \dot{v}_i \quad \text{or} \quad \nabla_{\mathbf{x}} \cdot \mathbf{\Sigma} + \rho \mathbf{b} = \rho \dot{\mathbf{v}} \tag{7.16}$$

(*c*) the energy equation (*5.32*),

$$\dot{u} = \frac{1}{\rho}\sigma_{ij}D_{ij} - \frac{1}{\rho}c_{i,i} + z \quad \text{or} \quad \dot{u} = \frac{1}{\rho}\mathbf{\Sigma} : \mathbf{D} - \frac{1}{\rho}\nabla_{\mathbf{x}} \cdot c + z \tag{7.17}$$

(*d*) the constitutive equations (*7.9*),

$$\sigma_{ij} = -p\delta_{ij} + \lambda^*\delta_{ij}D_{kk} + 2\mu^*D_{ij} \quad \text{or} \quad \mathbf{\Sigma} = -p\mathbf{I} + \lambda^*\mathbf{I}(\operatorname{tr}\mathbf{D}) + 2\mu^*\mathbf{D} \tag{7.18}$$

(*e*) the kinetic equation of state (*7.5*),

$$p = p(\rho, T) \tag{7.19}$$

If thermal effects are considered, as they very often must be in fluids problems, the additional equations

(*f*) the Fourier law of heat conduction (*6.71*),

$$c_i = -kT_{,i} \quad \text{or} \quad \mathbf{c} = -k\nabla T \tag{7.20}$$

(*g*) the caloric equation of state,

$$u = u(\rho, T) \tag{7.21}$$

are required. The system of equations (*7.15*) through (*7.21*) represents sixteen equations in sixteen unknowns and is therefore determinate.

If (*7.18*) above is substituted into (*7.16*) and the definition $2D_{ij} = (v_{i,j} + v_{j,i})$ is used, the equations that result from the combination are the *Navier-Stokes-Duhem equations of motion*,

$$\rho\dot{v}_i = \rho b_i - p_{,i} + (\lambda^* + \mu^*)v_{j,ji} + \mu^* v_{i,jj}$$

or

$$\rho\dot{\mathbf{v}} = \rho\mathbf{b} - \nabla p + (\lambda^* + \mu^*)\nabla(\nabla \cdot \mathbf{v}) + \mu^*\nabla^2\mathbf{v} \tag{7.22}$$

When the flow is incompressible $(v_{j,j} = 0)$, (*7.22*) reduce to the *Navier-Stokes equations* for *incompressible* flow,

$$\rho\dot{v}_i = \rho b_i - p_{,i} + \mu^* v_{i,jj} \quad \text{or} \quad \rho\dot{\mathbf{v}} = \rho\mathbf{b} - \nabla p + \mu^*\nabla^2\mathbf{v} \tag{7.23}$$

If Stokes condition is assumed $(\lambda^* = -\frac{2}{3}\mu^*)$, (*7.22*) reduce to the *Navier-Stokes* equations for *compressible* flow

$$\rho\dot{v}_i = \rho b_i - p_{,i} + \tfrac{1}{3}\mu^* v_{j,ji} + \mu^* v_{i,jj}$$

or

$$\rho\dot{\mathbf{v}} = \rho\mathbf{b} - \nabla p + \tfrac{1}{3}\mu^*\nabla(\nabla \cdot \mathbf{v}) + \mu^*\nabla^2\mathbf{v} \tag{7.24}$$

The Navier-Stokes equations (*7.23*), together with the continuity equation (*7.15*) form a complete set of four equations in four unknowns: the pressure $p$ and the three velocity components $v_i$. In any given problem, the solutions of this set of equations must satisfy boundary and initial conditions on traction and velocity components. For a viscous fluid, the appropriate boundary conditions at a fixed surface require both the normal and tangential components of velocity to vanish. This condition results from the experimentally established fact that a fluid adheres to and obtains the velocity of the boundary. For an inviscid fluid, only the normal velocity component is required to vanish on a fixed surface.

If the Navier-Stokes equations are put into dimensionless form, several ratios of the normalizing parameters appear. One of the most significant and commonly used ratios is the Reynolds number $N_{(R)}$ which expresses the ratio of inertia to viscous forces. Thus if a flow is characterized by a certain length $L$, velocity $V$ and density $\rho$, the Reynolds number is

$$N_{(R)} = VL/\nu \tag{7.25}$$

where $\nu = \mu^*/\rho$ is called the *kinematic viscosity*. For very large Reynolds numbers, the viscous contribution to the shear stress terms of the momentum equations may be neglected. In *turbulent flow*, the apparent stresses act on the time mean flow in a manner similar to the viscous stress effects in a *laminar flow*. If turbulence is not present, inertia effects outweigh viscous effects and the fluid behaves as though it were inviscid. The ability of a flow to support turbulent motions is related to the Reynolds number. It is only in the case of laminar flow that the constitutive relations (*7.18*) apply to real fluids.

## 7.4 STEADY FLOW. HYDROSTATICS. IRROTATIONAL FLOW

The motion of a fluid is referred to as a *steady flow* if the velocity components are independent of time. For this situation, the derivative $\partial v_i/\partial t$ is zero, and hence the material derivative of the velocity

$$\frac{dv_i}{dt} \equiv \dot{v}_i = \frac{\partial v_i}{\partial t} + v_j v_{i,j} \quad \text{or} \quad \frac{d\mathbf{v}}{dt} \equiv \dot{\mathbf{v}} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}}\mathbf{v} \tag{7.26}$$

reduces to the simple form

$$\dot{v}_i = v_j v_{i,j} \quad \text{or} \quad \dot{\mathbf{v}} = \mathbf{v}\cdot\nabla_{\mathbf{x}}\mathbf{v} \tag{7.27}$$

A steady flow in which the velocity is zero everywhere, causes the Navier-Stokes equations (*7.22*) to reduce to

$$\rho b_i = p_{,i} \quad \text{or} \quad \rho\mathbf{b} = \nabla_{\mathbf{x}} p \tag{7.28}$$

which describes the *hydrostatic equilibrium* situation. If the barotropic condition $\rho = \rho(p)$ is assumed, a *pressure function*

$$P(p) = \int_{p_0}^{p} \frac{dp}{\rho} \tag{7.29}$$

may be defined. Furthermore, if the body force may be prescribed by a potential function

$$b_i = -\Omega_{,i} \quad \text{or} \quad \mathbf{b} = -\nabla\Omega \tag{7.30}$$

equations (*7.28*) take on the form

$$(\Omega + P)_{,i} = 0 \quad \text{or} \quad \nabla(\Omega + P) = 0 \tag{7.31}$$

A flow in which the spin, or vorticity tensor (*4.21*),

$$V_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \quad \text{or} \quad \mathbf{V} = \tfrac{1}{2}(\mathbf{v}\nabla - \nabla\mathbf{v}) \tag{7.32}$$

vanishes everywhere is called an *irrotational flow*. The vorticity vector $q_i$ is related to the vorticity tensor by the equation

$$q_i = \epsilon_{ijk} V_{kj} \quad \text{or} \quad \mathbf{q} = \mathbf{V}_v \tag{7.33}$$

and therefore also vanishes for irrotational flow. Furthermore,

$$q_i = \epsilon_{ijk} v_{k,j} \quad \text{or} \quad \mathbf{q} = \nabla \times \mathbf{v} \tag{7.34}$$

and since $\nabla \times \mathbf{v} = 0$ is necessary and sufficient for a *velocity potential* $\phi$ to exist, the velocity vector for irrotational flow may be expressed by

$$v_i = -\phi_{,i} \quad \text{or} \quad \mathbf{v} = -\nabla\phi \tag{7.35}$$

## 7.5 PERFECT FLUIDS. BERNOULLI EQUATION. CIRCULATION

If the viscosity coefficients $\lambda^*$ and $\mu^*$ are zero, the resulting fluid is called an *inviscid* or *perfect* (frictionless) *fluid* and the Navier-Stokes-Duhem equations (*7.22*) reduce to the form

$$\rho\dot{v}_i = \rho b_i - p_{,i} \quad \text{or} \quad \rho\dot{\mathbf{v}} = \rho\mathbf{b} - \nabla p \tag{7.36}$$

which is known as the *Euler equation of motion*. For a barotropic fluid with conservative body forces, (*7.29*) and (*7.30*) may be introduced so that (*7.36*) becomes

$$\dot{v}_i = -(\Omega + P)_{,i} \quad \text{or} \quad \dot{\mathbf{v}} = -\nabla(\Omega + P) \tag{7.37}$$

For steady flow (*7.37*) may be written

$$v_j v_{i,j} = -(\Omega + P)_{,i} \quad \text{or} \quad \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla(\Omega + P) \tag{7.38}$$

If the Euler equation (*7.37*) is integrated along a streamline, the result is the well-known Bernoulli equation in the form (see Problem 7.17)

$$\Omega + P + v^2/2 + \int \frac{\partial v_i}{\partial t}\, dx_i = C(t) \tag{7.39}$$

For steady motion, $\partial v_i/\partial t = 0$ and $C(t)$ becomes the Bernoulli constant $C$ which is, in general, different along different streamlines. If the flow is irrotational as well, a single constant $C$ holds everywhere in the field of flow.

When the only body force present is gravity, the potential $\Omega = gh$ where $g$ is the gravitational constant and $h$ is the elevation above some reference level. Thus with $h_p = P/g$ defined as the *pressure* head, and $v^2/2g = h_v$ defined as the *velocity* head, Bernoulli's equation requires the total head along any streamline to be constant. For incompressible fluids (liquids), the equation takes the form

$$h + h_p + h_v = h + p/\rho g + v^2/2g = \text{constant} \tag{7.40}$$

By definition, the *velocity circulation* around a closed path of fluid particles is given by the line integral

$$\Gamma_c = \oint v_i dx_i \quad \text{or} \quad \Gamma_c = \oint \mathbf{v}\cdot d\mathbf{x} \tag{7.41}$$

From Stokes theorem (*1.153*) or (*1.154*), page 23, the line integral (*7.41*) may be converted to the surface integral

$$\Gamma_c = \int_S n_i \epsilon_{ijk} v_{k,j}\, dS \quad \text{or} \quad \Gamma_c = \int_S \hat{\mathbf{n}}\cdot(\nabla\times\mathbf{v})\, dS \tag{7.42}$$

where $\hat{\mathbf{n}}$ is the unit normal to the surface $S$ enclosed by the path. If the flow is irrotational, $\nabla \times \mathbf{v} = 0$ and the circulation is zero. In this case the integrand of (*7.41*) is the perfect differential $d\phi = -\mathbf{v}\cdot d\mathbf{x}$ with $\phi$ the velocity potential.

The material derivative $d\Gamma_c/dt$ of the circulation may be determined by using (*4.60*) which when applied to (*7.41*) gives

$$\dot{\Gamma}_c = \oint (\dot{v}_i\,dx_i + v_i\,dv_i) \quad \text{or} \quad \dot{\Gamma}_c = \oint (\dot{\mathbf{v}}\cdot d\mathbf{x} + \mathbf{v}\cdot d\mathbf{v}) \tag{7.43}$$

For a barotropic, inviscid fluid with conservative body forces the circulation may be shown to be a constant. This is known as *Kelvin's theorem* of constant circulation.

## 7.6 POTENTIAL FLOW. PLANE POTENTIAL FLOW

The term *potential flow* is often used to denote an irrotational flow since the condition of irrotationality, $\nabla \times \mathbf{v} = 0$, is necessary and sufficient for the existence of the velocity potential $\phi$ of (*7.35*). For a compressible irrotational flow, the Euler equation and the continuity equation may be linearized and combined as is done in acoustics to yield the governing wave equation

$$\ddot{\phi} = c^2\phi_{,ii} \quad \text{or} \quad \ddot{\phi} = c^2\nabla^2\phi \tag{7.44}$$

where $c$ is the velocity of sound in the fluid. For a steady irrotational flow of a compressible barotropic fluid, the Euler equation and continuity equation may be combined to give

$$(c^2\delta_{ij} - v_iv_j)v_{j,i} = 0 \quad \text{or} \quad c^2\nabla\cdot\mathbf{v} - \mathbf{v}\cdot(\mathbf{v}\cdot\nabla\mathbf{v}) = 0 \tag{7.45}$$

which is the so-called *gas dynamical equation.*

For incompressible potential flow the continuity equation attains the form

$$\phi_{,ii} = 0 \quad \text{or} \quad \nabla^2\phi = 0 \tag{7.46}$$

and solutions of this *Laplace equation* provide the velocity components through the definition (*7.35*). Boundary conditions on velocity must also be satisfied. On a fixed boundary, for example, $\partial\phi/\partial n = 0$. An important feature of this formulation rests in the fact that the Laplace equation is linear so that superposition of solutions is possible.

In a two-dimensional incompressible flow parallel to the $x_1x_2$ plane, $v_3 = 0$, and the continuity equation becomes

$$v_{\alpha,\alpha} = 0 \quad \text{or} \quad \nabla\cdot\mathbf{v} = 0 \tag{7.47}$$

where, as usual in this book, Greek subscripts have a range of two. By (*7.47*), regardless of whether the flow is irrotational or not, it is possible to introduce the *stream function* $\psi = \psi(x_1, x_2)$ such that

$$v_\alpha = -\epsilon_{\alpha\beta3}\psi_{,\beta} \tag{7.48}$$

If the plane flow is, indeed, irrotational so that

$$v_\alpha = -\phi_{,\alpha} \quad \text{or} \quad \mathbf{v} = -\nabla\phi \tag{7.49}$$

then from (*7.48*) and (*7.49*) the stream function and velocity potential are seen to satisfy the *Cauchy-Riemann* conditions

$$\phi_{,1} = \psi_{,2} \quad \text{and} \quad \phi_{,2} = -\psi_{,1} \tag{7.50}$$

By eliminating $\phi$ and $\psi$ in turn from (*7.50*) it is easily shown that

$$\phi_{,\alpha\alpha} = 0 \quad \text{or} \quad \nabla^2\phi = 0 \tag{7.51}$$

$$\psi_{,\alpha\alpha} = 0 \quad \text{or} \quad \nabla^2\psi = 0 \tag{7.52}$$

Thus both $\phi$ and $\psi$ are harmonic functions when the flow is irrotational. Furthermore, the complex potential

$$\Phi(z) = \phi(x_1, x_2) + i\psi(x_1, x_2) \tag{7.53}$$

is an analytic function of the complex variable, $z = x_1 + ix_2$ so that its derivative $d\Psi/dz$ defines the *complex velocity*

$$d\Psi/dz = -v_1 + iv_2 \tag{7.54}$$

# Solved Problems

## FUNDAMENTALS OF FLUIDS. NEWTONIAN FLUIDS (Sec. 7.1-7.3)

**7.1.** Show that the deviator $s_{ij}$ for the stress tensor $\sigma_{ij}$ of (*7.3*) is equal to $t_{ij}$, the deviator of $\tau_{ij}$ of (*7.3*).

From (*7.3*), $\sigma_{ii} = -3p + \tau_{ii}$ and so here

$$s_{ij} = \sigma_{ij} - \delta_{ij}\sigma_{kk}/3 = -p\delta_{ij} + \tau_{ij} - \delta_{ij}(-3p + \tau_{kk})/3 = \tau_{ij} - \delta_{ij}\tau_{kk}/3 = t_{ij}$$

**7.2.** Determine the mean normal stress $\sigma_{ii}/3$ for an incompressible Stokesian (nonlinear) fluid for which $\tau_{ij} = \alpha D_{ij} + \beta D_{ik}D_{kj}$ where $\alpha$ and $\beta$ are constants.

From (*7.3*), $\sigma_{ij} = -p\delta_{ij} + \alpha D_{ij} + \beta D_{ik}D_{kj}$ and so $\sigma_{ii} = -3p + \alpha D_{ii} + \beta D_{ik}D_{ki}$. But $D_{ik} = D_{ki}$ and $D_{ii} = v_{i,i} = 0$ for an incompressible fluid so that

$$\sigma_{ii}/3 = -p + \beta D_{ij}D_{ij}/3 = -p - 2\beta \text{II}_D/3$$

where $\text{II}_D$ is the second invariant of the rate of deformation tensor.

**7.3.** Frictionless adiabatic, or isentropic flow of an ideal gas, is a barotropic flow for which $p = c\rho^k$ where $C$ and $k$ are constants with $k = c^{(p)}/c^{(v)}$, the ratio of specific heat at constant pressure to that at constant volume. Determine the temperature-density and temperature-pressure relationships for such a flow.

Inserting $p = C\rho^k$ into equation (*7.6*), the temperature-density relationship is $\rho^{k-1}/T = R/C$, a constant. Also, since $\rho = (p/C)^{1/k}$ here, (*7.6*) yields the temperature-pressure relationship as $p^{(k-1)/k}/T = R/C^{1/k}$, a constant.

**7.4.** Determine the constitutive equation for a Newtonian fluid with zero bulk viscosity, i.e. with $\kappa^* \equiv 0$.

If $\kappa^* \equiv 0$, $\lambda^* = -2\mu^*/3$ by (*7.11*) and so (*7.9*) becomes $\sigma_{ij} = -p\delta_{ij} - (2\mu^*/3)\delta_{ij}D_{kk} + 2\mu^* D_{ij}$ which is expressed in terms of the rate of deformation deviator by

$$\sigma_{ij} = -p\delta_{ij} + 2\mu^*(D_{ij} - \delta_{ij}D_{kk}/3) = -p\delta_{ij} + 2\mu^* D'_{ij}$$

If the deviator stress $s_{ij}$ is introduced, this constitutive relation is given by the two equations $s_{ij} = 2\mu^* D'_{ij}$ and $\sigma_{ii} = -3p$.

**7.5.** Determine an expression for the "stress power" $\sigma_{ij}D_{ij}$ of a Newtonian fluid having equation (*7.9*) as its constitutive relation.

From (*7.9*) and the stress power definition,

$$\sigma_{ij}D_{ij} = -p\delta_{ij}D_{ij} + \lambda^*\delta_{ij}D_{kk}D_{ij} + 2\mu^*D_{ij}D_{ij} = -pD_{ii} + \lambda^*D_{ii}D_{jj} + 2\mu^*D_{ij}D_{ij}$$

In symbolic notation, this expression is written

$$\mathbf{\Sigma}:\mathbf{D} = -p(\operatorname{tr}\mathbf{D}) + \lambda^*(\operatorname{tr}\mathbf{D})^2 + 2\mu^*\mathbf{D}:\mathbf{D}$$

In terms of $D'_{ij}$ the expression is

$$\sigma_{ij}D_{ij} = -pD_{ii} + \lambda^*D_{ii}D_{jj} + 2\mu^*(D'_{ij} + \delta_{ij}D_{kk}/3)(D'_{ij} + \delta_{ij}D_{qq}/3) = -pD_{ii} + \kappa^*D_{ii}D_{jj} + 2\mu^*D'_{ij}D'_{ij}$$

In symbolic notation,

$$\mathbf{\Sigma}:\mathbf{D} = -p(\operatorname{tr}\mathbf{D}) + \kappa^*(\operatorname{tr}\mathbf{D})^2 + 2\mu^*\mathbf{D}':\mathbf{D}'$$

**7.6.** Determine the conditions under which the mean normal pressure $p_{(m)} = -\sigma_{ii}/3$ is equal to the thermodynamic pressure $p$ for a Newtonian fluid.

With the constitutive equations in the form (*7.13*) and (*7.14*), the latter equation gives $p_{(m)} - p = -\kappa^*D_{ii}$. Thus $p_{(m)} = p$ when $\kappa^* = 0$ (by (*7.11*) when $\lambda^* = -\frac{2}{3}\mu^*$) or when $D_{ii} = 0$.

**7.7.** Verify the Navier-Stokes-Duhem equations of motion (*7.22*) for a Newtonian fluid and determine the form of the energy equation (*7.17*) for this fluid if the heat conduction follows the Fourier law (*7.20*).

Since $D_{ii} = v_{i,i}$, equation (*7.18*) may be written $\sigma_{ij} = -p\delta_{ij} + \lambda^*\delta_{ij}v_{k,k} + \mu^*(v_{i,j} + v_{j,i})$. Thus

$$\sigma_{ij,j} = -p_{,j}\delta_{ij} + \lambda^*\delta_{ij}v_{k,kj} + \mu^*(v_{i,jj} + v_{j,ij}) = -p_{,i} + (\lambda^* + \mu^*)v_{j,ji} + \mu^*v_{i,jj}$$

and with this expression inserted into (*7.16*) a direct verification of (*7.22*) is complete.

Substituting the above equation for $\sigma_{ij}$ together with (*7.20*) into the energy equation (*7.17*), the result is

$$\rho\dot{u} = [-p\delta_{ij} + \lambda^*\delta_{ij}v_{k,k} + \mu^*(v_{i,j} + v_{j,i})](v_{i,j} + v_{j,i})/2 - kT_{,ii} + \rho z$$

which reduces to

$$\rho\dot{u} = -pv_{i,i} + \lambda^*v_{i,i}v_{j,j} + \mu^*(v_{i,j} + v_{j,i})(v_{i,j} + v_{j,i})/2 - kT_{,ii} + \rho z$$

**7.8.** Determine the traction force $T_i$ acting on the closed surface $S$ which surrounds the volume $V$ of a Newtonian fluid for which the bulk viscosity is zero.

The element of traction is $dT_i = t_i^{(\hat{n})}\,dS$ and the total traction force is $T_i = \int_S t_i^{(\hat{n})}\,dS$ which because of the stress principle is $T_i = \int_S \sigma_{ji}n_j\,dS$. From Problem 7.4, this becomes

$$T_i = \int_S (-p\delta_{ij} + 2\mu^*D'_{ij})n_j\,dS$$

for a zero bulk modulus fluid; and upon application of Gauss' theorem,

$$T_i = \int_V (2\mu^*D'_{ij,j} - p_{,i})\,dV$$

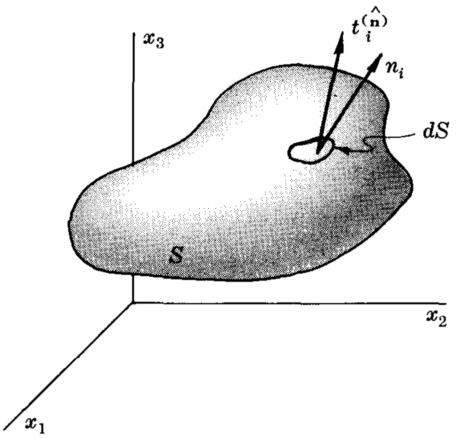



Fig. 7-1

**7.9.** **In an axisymmetric flow along the $x_3$ axis the velocity is taken as a function of $x_3$ and $r$ where $r^2 = x_1^2 + x_2^2$. If the velocity is expressed by $\mathbf{v} = q\hat{\mathbf{e}}_r + v_3\hat{\mathbf{e}}_3$ where $\hat{\mathbf{e}}_r$ is the unit radial vector, determine the form of the continuity equation.**

Equation *(5.4)* gives the continuity equation in symbolic notation as $\partial\rho/\partial t + \nabla \cdot (\rho\mathbf{v}) = 0$. Here the cylindrical form of the operator $\nabla$ may be used to give $\nabla \cdot (\rho\mathbf{v}) = \dfrac{1}{r}\dfrac{\partial(r\rho q)}{\partial r} + \dfrac{\partial(\rho v_3)}{\partial x_3}$. Inserting this into *(5.4)* and simplifying, the continuity equation becomes

$$r(\partial\rho/\partial t) + \partial(r\rho q)/\partial r + \partial(r\rho v_3)/\partial x_3 = 0$$

**7.10.** **In a two-dimensional flow parallel to the $x_1x_2$ plane, $v_3$ and $\partial/\partial x_3$ are zero. Determine the Navier-Stokes equations for an incompressible fluid and the form of the continuity equation for this case.**

From *(7.23)* with $i = 3$, $\rho b_3 = p_{,3}$ and when $i = 1, 2$, $\rho\dot{v}_\alpha = \rho b_\alpha - p_{,\alpha} + \mu^* v_{\alpha,\beta\beta}$. The continuity equation *(7.15)* reduces to $v_{\alpha,\alpha} = 0$.

If body forces were zero and $v_1 = v_1(x_1, x_2, t)$, $v_2 = 0$, $p = p(x_1, x_2, t)$ the necessary equations would be $\rho\dot{v}_1 = -\partial p/\partial x_1 + \mu^*(\partial^2 v_1/\partial x_1^2 + \partial^2 v_1/\partial x_2^2)$ and $\partial v_1/\partial x_1 = 0$.

## HYDROSTATICS. STEADY AND IRROTATIONAL FLOW (Sec. 7.4)

**7.11.** **Assuming air is an ideal gas whose temperature varies linearly with altitude as $T = T_0 - \alpha x_3$ where $T_0$ is ground level temperature and $x_3$ measures height above the earth, determine the air pressure in the atmosphere as a function of $x_3$ under hydrostatic conditions.**

From *(7.6)* in this case, $p = \rho R(T_0 - \alpha x_3)$; and from *(7.28)* with the body force $b_3 = -g$, the gravitational constant, $dp/dx_3 = -\rho g = -pg/R(T_0 - \alpha x_3)$. Separating variables and integrating yields $\ln p = (g/R\alpha)\ln(T_0 - \alpha x_3) + \ln C$ where $C$ is a constant of integration. Thus $p = C(T_0 - \alpha x_3)^{g/R\alpha}$ and if $p = p_0$ when $x_3 = 0$, $C = p_0 T_0^{-g/R\alpha}$ and so $p = p_0(1 - \alpha x_3/T_0)^{g/R\alpha}$.

**7.12.** **A barotropic fluid having the equation of state $p = \lambda\rho^k$ where $\lambda$ and $k$ are constants is at rest in a gravity field in the $x_3$ direction. Determine the pressure in the fluid with respect to $x_3$ and $p_0$, the pressure at $x_3 = 0$.**

From *(7.28)*, $dp/dx_3 = -\rho g$, $dp/dx_1 = dp/dx_2 = 0$. Note that pressure in $x_1$ and $x_2$ directions is constant in the absence of body forces $b_1$ and $b_2$. Since here $\rho = (p/\lambda)^{1/k}$, $p^{-1/k}dp = -g\lambda^{-1/k}dx_3$ and integration gives $(k/(k-1))p^{(k-1)/k} = -g\lambda^{-1/k}x_3 + C$. But $p = p_0$ when $x_3 = 0$ so that $C = (k/(k-1))p_0^{(k-1)/k}$. Therefore $x_3 = (kp_0/(k-1)g\rho_0)(1 - (p/p_0)^{(k-1)/k})$ where $\rho_0 = (p_0/\lambda)^{1/k}$.

**7.13.** **A large container filled with an incompressible liquid is accelerated at a constant rate $\mathbf{a} = a_2\hat{\mathbf{e}}_2 + a_3\hat{\mathbf{e}}_3$ in a gravity field which is parallel to the $x_3$ direction. Determine the slope of the free surface of the liquid.**

From *(7.28)*, $dp/dx_1 = 0$, $dp/dx_2 = \rho a_2$ and $dp/dx_3 = -\rho(g - a_3)$. Integrating, $p = \rho a_2 x_2 + f(x_3)$ and $p = -\rho(g - a_3)x_3 + h(x_2)$ where $f$ and $h$ are arbitrary functions of their arguments. In general, therefore, $p = \rho a_2 x_2 - \rho(g - a_3)x_3 + p_0$ where $p_0$ is the pressure at the origin of coordinates on the free surface. Since $p = p_0$ everywhere on the free surface, the equation of that surface is $x_2/x_3 = (g - a_3)/a_2$.

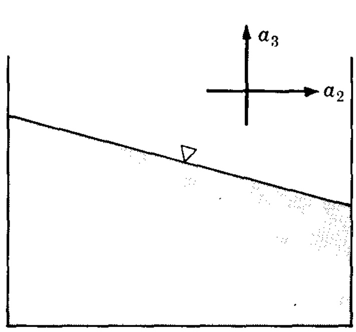


Fig. 7-2

**7.14.** If a fluid motion is very slow so that higher order terms in the velocity are negligible, a limiting case known as *creeping flow* results. For this case show that in a steady incompressible flow with zero body forces the pressure is a harmonic function, i.e. $\nabla^2 p = 0$.

For incompressible flow the Navier-Stokes equations (*7.23*) are

$$\rho(\partial v_i/\partial t + v_j v_{i,j}) = \rho b_i - p_{,i} + \mu^* v_{i,jj}$$

and for creeping flow these linearize to the form

$$\rho(\partial v_i/\partial t) = \rho b_i - p_{,i} + \mu^* v_{i,jj}$$

Hence for steady flow with zero body forces, $p_{,i} = \mu^* v_{i,jj}$. Taking the divergence of this equation yields $p_{,ii} = \mu^* v_{i,ijj}$; and since the continuity equation for incompressible flow is $v_{i,i} = 0$, it follows that here $p_{,ii} = \nabla^2 p = 0$.

**7.15.** Express the continuity equation and the Navier-Stokes-Duhem equations in terms of the velocity potential $\phi$ for an irrotational motion.

By (*7.35*), $v_i = -\phi_{,i}$ so that from (*7.15*) the continuity equation becomes $\dot{\rho} - \rho\nabla^2\phi = 0$. Also with $v_i = -\phi_{,i}$, (*7.22*) becomes

$$-\rho\dot{\phi}_{,i} = \rho b_i - p_{,i} - (\lambda^* + \mu^*)\phi_{,jji} - \mu^*\phi_{,ijj}$$

or

$$-\rho(\partial\phi_{,i}/\partial t + \phi_{,k}\phi_{,ik}) = \rho b_i - p_{,i} - (\lambda^* + 2\mu^*)\phi_{,jji}$$

In symbolic notation this equation is written

$$-\rho\nabla(\partial\phi/\partial t + (\nabla\phi)^2/2) = \rho\mathbf{b} - \nabla p - (\lambda^* + 2\mu^*)\nabla(\nabla^2\phi)$$

**7.16.** Determine the pressure function $P(p)$ for a barotropic fluid having the equation of state $p = \lambda\rho^k$ where $\lambda$ and $k$ are constants.

From the definition (*7.29*),

$$P(p) = \int_{p_0}^{p} \frac{dp}{\rho} = \int_{p_0}^{p} (p/\lambda)^{-1/k}\, dp = \frac{k\lambda^{1/k}}{k-1}\Big[p^{(k-1)/k}\Big]_{p_0}^{p} = \frac{k}{k-1}\left(\frac{p}{\rho} - \frac{p_0}{\rho_0}\right)$$

Also since $dp = \lambda k \rho^{k-1}\, d\rho$, the same result may be obtained from

$$P(p) = \int_{\rho_0}^{\rho} \lambda k \rho^{k-2}\, d\rho = \frac{\lambda k}{k-1}\Big[\rho^{k-1}\Big]_{\rho_0}^{\rho} = \frac{k}{k-1}\left(\frac{p}{\rho} - \frac{p_0}{\rho_0}\right)$$

## PERFECT FLUIDS. BERNOULLI EQUATION. CIRCULATION (Sec. 7.5)

**7.17.** Derive equation (*7.39*) by integrating Euler's equation (*7.37*) along a streamline.

Let $dx_i$ be an increment of displacement along a streamline. Taking the scalar product of this increment with (*7.37*) and integrating gives

$$\int \frac{\partial v_i}{\partial t}\, dx_i + \int v_j v_{i,j}\, dx_i + \int \Omega_{,i}\, dx_i + \int P_{,i}\, dx_i = C(t)$$

Since $\Omega_{,i}\, dx_i = d\Omega$ and $P_{,i}\, dx_i = dP$ the last two terms integrate at once. Also, along a streamline, $dx_i = (v_i/v)\, ds$ where $ds$ is the increment of distance. Thus in the second integral,

$$v_j v_{i,j}\, dx_i = v_j v_{i,j}(v_i/v)\, ds = v_i v_{i,j}(v_j/v)\, ds = v_i v_{i,j}\, dx_j = v_i\, dv_i$$

Therefore $\int v_j v_{i,j}\, dx_i = \int v_i\, dv_i = \frac{1}{2}v_i v_i = \frac{1}{2}v^2$, and (*7.39*) is achieved.

**7.18.** **The barotropic fluid of Problem 7.16 flows from a large closed tank through a thin smooth pipe. If the pressure in the tank is $N$ times the atmospheric pressure, determine the speed of the emerging fluid.**

Applying Bernoulli's equation for steady flow between point $A$, at rest in fluid of the tank and point $B$, in the emerging free stream, (*7.39*) assumes the form $\Omega_A + P_A + \frac{1}{2}v_A^2 = \Omega_B + P_B + \frac{1}{2}v_B^2$. But $v_A = 0$, and if gravity is assumed negligible this equation becomes (see Problem 7.16),

$$\frac{k}{k-1}\left(\frac{p_A}{\rho_A} - \frac{p_B}{\rho_B}\right) = \tfrac{1}{2}v_B^2 \qquad \text{or} \qquad v_B^2 = \frac{2k}{k-1}\frac{p_B}{\rho_B}\left(\frac{N\rho_B}{\rho_A} - 1\right)$$

Since $\rho_B/\rho_A = (p_B/p_A)^{-1/k} = N^{-1/k}$ the result may be written

$$v_B^2 = \frac{2k}{k-1}\frac{p_B}{\rho_B}(N^{(k-1)/k} - 1)$$

**7.19.** **Show that for a barotropic, inviscid fluid with conservative body forces the rate of change of the circulation is zero (Kelvin's theorem).**

From (*7.43*) $\dot{\Gamma}_c = \oint (\dot{v}_i\, dx_i + v_i\, dv_i)$ and by (*7.37*), $\dot{v}_i = -(\Omega + P)_{,i}$ for the case at hand. Thus $\dot{\Gamma}_c = \oint (-\Omega_{,i}\, dx_i - P_{,i}\, dx_i + v_i\, dv_i) = -\oint (d\Omega + dP - d(v^2/2)) = -\oint d(\Omega + P - v^2/2) = 0$, the integrand being a perfect differential.

**7.20.** **Determine the circulation around the square $x_1 = \pm 1$, $x_2 = \pm 1$, $x_3 = 0$ (see Fig. 7-3) for the two-dimensional flow $\mathbf{v} = (x_1 + x_2)\hat{\mathbf{e}}_1 + (x_1^2 - x_2)\hat{\mathbf{e}}_2$.**

Using the symbolic form of (*7.42*) with $\hat{\mathbf{n}} = \hat{\mathbf{e}}_3$ and $\nabla \times \mathbf{v} = (2x_1 - 1)\,\hat{\mathbf{e}}_3$,

$$\Gamma_c = \int_{-1}^{1}\int_{-1}^{1} (2x_1 - 1)\, dx_1\, dx_2 = -4$$

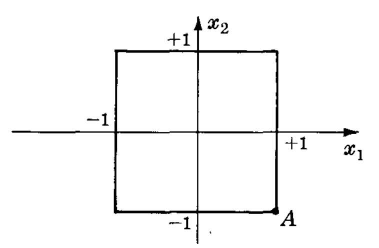



Fig. 7-3

The same result is obtained from (*7.41*) where

$$\Gamma_c = \oint \mathbf{v} \cdot d\mathbf{x}$$

$$= \int_{-1}^{1} (1 - x_2)\, dx_2 + \int_{1}^{-1} (x_1 + 1)\, dx_1 + \int_{1}^{-1} (1 - x_2)\, dx_2 + \int_{-1}^{1} (x_1 - 1)\, dx_1 = -4$$

with the integration proceeding counterclockwise from $A$.

## POTENTIAL FLOW. PLANE POTENTIAL FLOW (Sec. 7.6)

**7.21.** **Give the derivation of the gas dynamical equation (*7.45*) and express this equation in terms of the velocity potential $\phi$.**

For a steady flow the continuity equation (*5.4*) becomes $\rho_{,i}v_i + \rho v_{i,i} = 0$ and the Euler equation (*7.36*) becomes $\rho v_j v_{i,j} + p_{,i} = 0$ if body forces are neglected. For a barotropic fluid, $p = p(\rho)$ and so $dp = (\partial p/\partial x_i)(\partial x_i/\partial \rho)d\rho$; or rearranging, $p_{,i} = (dp/d\rho)\rho_{,i} = c^2\rho_{,i}$ where $c$ is the local velocity of sound. Inserting this into the Euler equation and multiplying by $v_i$ gives $\rho v_i v_j v_{i,j} + c^2 v_i \rho_{,i} = 0$. From the continuity equation $c^2 v_i \rho_{,i} = -c^2 v_{i,i} = -c^2\delta_{ij}v_{i,j}$ and so $(c^2\delta_{ij} - v_i v_j)v_{i,j} = 0$. In terms of $\phi_{,i} = -v_i$ this becomes $(c^2\delta_{ij} - \phi_{,i}\phi_{,j})\phi_{,ij} = 0$.

**7.22.** **Show that the function $\phi = A(-x_1^2 - x_2^2 + 2x_3^2)$ satisfies the Laplace equation and determine the resulting velocity components.**

Substituting $\phi$ into (*7.46*) gives $-2A - 2A + 4A \equiv 0$. From (*7.35*), $v_1 = 2Ax_1$, $v_2 = 2Ax_2$, $v_3 = -4Ax_3$. Also, by the analysis of Problem 4.7 the streamlines in the $x_1$ plane are represented by $x_2^2 x_3 = \text{constant}$; in the $x_2$ plane by $x_1^2 x_3 = \text{constant}$. Thus the flow is in along the $x_3$ axis against the $x_1x_2$ plane (fixed wall).

**7.23.** Show that the stream function $\psi(x_1, x_2)$ is constant along any streamline.

From (7.48) and the differential equation of a streamline, $dx_1/v_1 = dx_2/v_2$ (see Problem 4.7), $-dx_1/\psi_{,2} = dx_2/\psi_{,1}$ or $\psi_{,1}dx_1 + \psi_{,2}dx_2 = d\psi = 0$. Thus $\psi$ = a constant along any streamline.

**7.24.** Verify that $\phi = A(x_1^2 - x_2^2)$ is a valid velocity potential and describe the flow field.

For the given $\phi$, (7.46) is satisfied identically by $2A - 2A = 0$; and from (7.49), $v_1 = -2Ax_1$, $v_2 = +2Ax_2$. The streamlines are determined by integrating $dx_1/x_1 = -dx_2/x_2$ to give the rectangular hyperbolas $x_1x_2 = C$ (Fig. 7-4). The equipotential lines $A(x_1^2 - x_2^2) = C_1$ form an orthogonal set of rectangular hyperbolas with the streamlines. Finally from (7.50), $\psi = -2Ax_1x_2 + C_0$ and is seen to be constant along the streamlines as was asserted in Problem 7.23.

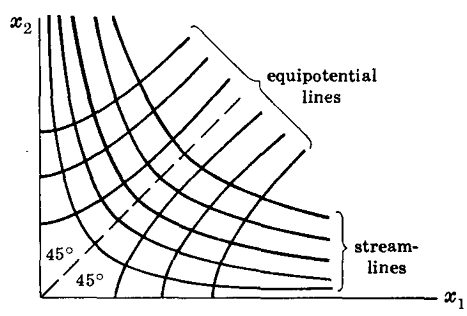


Fig. 7-4

**7.25.** A velocity potential is given by $\phi = Ax_1 + Bx_1/r^2$ where $r^2 = x_1^2 + x_2^2$. Determine the stream function $\psi$ for this flow.

From (7.50), $\psi_{,1} = -\phi_{,2} = 2Bx_1x_2/r^4$ so that by integrating, $\psi = -Bx_2/r^2 + f(x_2)$ where $f(x_2)$ is an arbitrary function of $x_2$. Differentiating, $\psi_{,2} = -B(x_1^2 - x_2^2)/r^4 + f'(x_2)$. But from (7.50), $\psi_{,2} = \phi_{,1} = A + B(-x_1^2 + x_2^2)/r^4$. Thus $f'(x_2) = A$ and $f(x_2) = Ax_2 + C$. Finally then $\psi = Ax_2 - Bx_2/r^2 + C$.

**7.26.** Differentiate the complex potential $\Phi(z) = A/z$ to obtain the velocity components.

Here $d\Phi/dz = -A/z^2 = -A/(x_1 + ix_2)^2$ which after some algebra becomes $d\Phi/dz = -A(x_1^2 - x_2^2)/r^4 + i2Ax_1x_2/r^4$. Thus

$$v_1 = A(x_1^2 - x_2^2)r^4 \quad \text{and} \quad v_2 = 2Ax_1x_2/r^4$$

Note that since $\Phi = A/z = A(x_1 - ix_2)/r^2$, $\phi = Ax_1/r^2$ and $\psi = -Ax_2/r^2$. Also note that

$$v_1 = -\phi_{,1} = A(x_1^2 - x_2^2)/r^4 \quad \text{and} \quad v_2 = -\phi_{,2} = 2Ax_1x_2/r^4$$

## MISCELLANEOUS PROBLEMS

**7.27.** Derive the one-dimensional continuity equation for the flow of an inviscid incompressible fluid through a stream tube.

Let $V$ be the volume between arbitrary cross sections $A$ and $B$ of the stream tube shown in Fig. 7-5. In integral form, for this volume (5.2) becomes $\int_V \nabla \cdot \mathbf{v}\, dV = 0$ since $\rho$ is constant here. Converting by Gauss' theorem, $\int_S \hat{\mathbf{n}} \cdot \mathbf{v}\, dS = 0$ where $\hat{\mathbf{n}}$ is the outward unit normal to the surface $S$ enclosing $V$. Since $\hat{\mathbf{n}} \perp \mathbf{v}$ on the lateral surface, the integration reduces to

Fig. 7-5

$$\int_{S_A} \hat{\mathbf{n}}_A \cdot \mathbf{v}_A\, dS + \int_{S_B} \hat{\mathbf{n}}_B \cdot \mathbf{v}_B\, dS = 0$$

The velocity is assumed uniform and perpendicular over $S_A$ and $S_B$; and since $\mathbf{v}_B = -v_B\hat{\mathbf{n}}_B$, $v_A\int_{S_A} dS - v_B\int_{S_B} dS = 0$ or $v_A S_A = v_B S_B$ = a constant.

**7.28.** The stress tensor at a given point for a Newtonian fluid with zero bulk viscosity is $\sigma_{ij} = \begin{pmatrix} -6 & 2 & -1 \\ 2 & -9 & 4 \\ -1 & 4 & -3 \end{pmatrix}$. Determine $\tau_{ij}$.

From (*7.14*), for this fluid $p = -\sigma_{ii}/3 = 6$. Then from (*7.3*),

$$\tau_{ij} = \sigma_{ij} + 6\delta_{ij} \quad \text{or} \quad \begin{pmatrix} -6 & 2 & -1 \\ 2 & -9 & 4 \\ -1 & 4 & -3 \end{pmatrix} + \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = \begin{pmatrix} 0 & 2 & -1 \\ 2 & -3 & 4 \\ -1 & 4 & 3 \end{pmatrix}$$

**7.29.** Show that $\sigma_{ij}$ and $\tau_{ij}$ of (*7.3*) have the same principal axes.

When written out, (*7.3*) becomes $\sigma_{11} = -p + \tau_{11}$, $\sigma_{22} = -p + \tau_{22}$, $\sigma_{33} = -p + \tau_{33}$, $\sigma_{12} = \tau_{12}$, $\sigma_{23} = \tau_{23}$, $\sigma_{13} = \tau_{13}$. For principal directions $x_i^*$ of $\sigma_{ij}$, $\sigma_{12}^* = \sigma_{23}^* = \sigma_{13}^* = 0$ and by the last three equations of (*7.3*), $\sigma_{ij}^* = \tau_{ij}^* = 0$ for $i \neq j$. Thus $x_i^*$ are principal axes for $\tau_{ij}$ also.

**7.30.** A dissipation potential $\Phi_D$ is often defined for a Newtonian fluid by the relationship $\Phi_D = (\kappa/2)D_{ii}D_{jj} + \mu^* D'_{ij}D'_{ij}$. Show that $\partial\Phi_D/\partial D_{ij} = \tau_{ij}$.

Here $\partial\Phi_D/\partial D_{pq} = (\kappa/2)[D_{ii}(\partial D_{jj}/\partial D_{pq}) + (\partial D_{ii}/\partial D_{pq})D_{jj}] + 2\mu^*[D'_{ij}(\partial D'_{ij}/\partial D_{pq})]$. But $\partial D_{ii}/\partial D_{pq} = \delta_{ip}\delta_{iq} = \delta_{pq}$ and $\partial D'_{ij}/\partial D_{pq} = \delta_{ip}\delta_{jq} - \delta_{ij}\delta_{pq}/3$ so that

$$\partial\Phi_D/\partial D_{pq} = \kappa D_{ii}\delta_{pq} + 2\mu^*(D_{ij} - \delta_{ij}D_{kk}/3)(\delta_{ip}\delta_{jq} - \delta_{ij}\delta_{pq}/3) = \kappa D_{ii}\delta_{pq} + 2\mu^*(D_{pq} - \delta_{pq}D_{ii}/3)$$

Finally since $\kappa = \lambda^* + 2\mu^*/3$,

$$\partial\Phi_D/\partial D_{pq} = \lambda^*\delta_{pq}D_{ii} + 2\mu^* D_{pq} = \tau_{pq}$$

**7.31.** Determine the pressure-density relationship for the ideal gas discussed in Problem 7.11.

At $x_3 = 0$, $\rho = \rho_0$ and $p = p_0$. The ideal gas law (*7.6*) is here $p = \rho R(T_0 - \alpha x_3)$ so that $p_0 = \rho_0 R T_0$; and from the pressure elevation relationship $p = p_0(1 - \alpha x_3/T_0)^{g/R\alpha}$ of Problem 7.11, $\rho/\rho_0 = (T/T_0)^{(g/R\alpha - 1)}$. Thus writing $p = p_0(1 - \alpha x_3/T_0)^{g/R\alpha}$ in the form $p/p_0 = (T/T_0)^{g/R\alpha}$, it is seen that $T/T_0 = (p/p_0)^{+R\alpha/g}$ and so $\rho/\rho_0 = (p/p_0)^{(1 - R\alpha/g)}$.

**7.32.** For a barotropic inviscid fluid with conservative body forces show that the material derivative of the total vorticity, $\dfrac{d}{dt}\displaystyle\int_V q_i\,dV = \int_S v_i q_j\,dS_j$.

From (*4.54*) and the results of Problem 4.33, $\dfrac{d}{dt}\displaystyle\int_V q_i\,dV = \int_S (\epsilon_{ijk}a_k + q_j v_i)\,dS_j$. But here $a_k = -(\Omega + P)_{,k}$ from (*7.37*); and by the divergence theorem (*1.157*),

$$\int_S \epsilon_{ijk}(\Omega + P)_{,k}\,dS_j = \int_V \epsilon_{ijk}(\Omega + P)_{,kj}\,dV = 0$$

since the integrand is zero (product of a symmetric and antisymmetric tensor). Hence

$$\frac{d}{dt}\int_V q_i\,dV = \int_S q_j v_i\,dS_j$$

**7.33.** For an incompressible Newtonian fluid moving inside a closed rigid container at rest, show that the time rate of change of kinetic energy of the fluid is $-\mu^*\displaystyle\int_V q^2\,dV$ assuming zero body forces. $q$ is the magnitude of the vorticity vector.

From Problem 5.27, the time rate of change of kinetic energy of a continuum is

$$\frac{dK}{dt} = \int_V \rho b_i v_i \, dV - \int_V \sigma_{ij} v_{i,j} \, dV + \int_S v_i t_i^{(\hat{n})} \, dS$$

In this problem the first and third integrals are zero; and for a Newtonian fluid by (*7.18*),

$$\frac{dK}{dt} = -\int_V \sigma_{ij} v_{i,j} \, dV = -\int_V (-p\delta_{ij} + \lambda^* \delta_{ij} D_{kk} + 2\mu^* D_{ij}) v_{i,j} \, dV$$

But incompressibility means $v_{i,i} = D_{ii} = 0$ and so

$$\frac{dK}{dt} = -2\mu^* \int_V D_{ij} v_{i,j} \, dV = -\mu^* \int_V (v_{i,j} + v_{j,i}) v_{i,j} \, dV$$

$$= -\mu^* \int_V (\epsilon_{kji} q_k) v_{i,j} \, dV = -\mu^* \int_V q_k (\epsilon_{kji} v_{i,j}) \, dV = -\mu^* \int_V q_k q_k \, dV$$

**7.34.** Show that for a perfect fluid with negligible body forces the rate of change of circulation $\dot{\Gamma}_c$ may be given by $-\int_S \epsilon_{ijk} (1/\rho)_{,j} p_{,k} \, dS_i$.

From (*7.43*), $\dot{\Gamma}_c = \oint \dot{v}_i \, dx_i + \oint v_i \, dv_i$; and since $\oint d(\frac{1}{2}v^2) = 0$, the second integral is zero. From (*7.36*) with $b_i = 0$, $\dot{v}_i = -p_{,i}/\rho$ and so now

$$\dot{\Gamma}_c = -\oint (p_{,k}/\rho) \, dx_k = -\int_S \epsilon_{ijk} (p_{,k}/\rho)_{,j} n_i \, dS$$

where (*7.42*) has been used in converting to the surface integral. Differentiating as indicated,

$$\dot{\Gamma}_c = -\int_S \epsilon_{ijk} [(1/\rho)_{,j} p_{,k} + p_{,kj}/\rho] \, dS_i = -\int_S \epsilon_{ijk} (1/\rho)_{,j} p_{,k} \, dS_i$$

# Supplementary Problems

**7.35.** The constitutive equation for an isotropic fluid is given by $\sigma_{ij} = -p\delta_{ij} + K_{ijpq} D_{pq}$ with $K_{ijpq}$ constants independent of the coordinates. Show that the principal axes of stress and rate of deformation coincide.

**7.36.** Show that $(1/\rho)(d\rho/dt) = 0$ is a condition for $-\sigma_{ii}/3 = p$ for a Newtonian fluid.

**7.37.** Show that the constitutive relations for a Newtonian fluid with zero bulk viscosity may be expressed by the pair of equations $s_{ij} = 2\mu^* D_{ij}$ and $-\sigma_{ii} = 3p$.

**7.38.** Show that in terms of the vorticity vector $\mathbf{q}$ the Navier-Stokes equations may be written $\dot{\mathbf{v}} = \mathbf{b} - \nabla p/\rho - \nu^* \nabla \times \mathbf{q}$ where $\nu^* = \mu^*/\rho$ is the kinematic viscosity. Show that for irrotational motion this equation reduces to (*7.36*).

**7.39.** If a fluid moves radially with the velocity $v = v(r, t)$ where $r^2 = x_i x_i$, show that the equation of continuity is $\dfrac{\partial \rho}{\partial t} + v \dfrac{\partial \rho}{\partial r} + \dfrac{\rho}{r^2} \dfrac{\partial}{\partial r}(r^2 v) = 0$.

**7.40.** A liquid rotates as a rigid body with constant angular velocity $\omega$ about the vertical $x_3$ axis. If gravity is the only body force, show that $p/\rho - \omega^2 r^2/2 + g x_3 = \text{constant}$.

**7.41.** For an ideal gas under isothermal conditions (constant temperature $= T_0$), show that $\rho/\rho_0 = p/p_0 = e^{-(g/RT_0 x_3)}$ where $\rho_0$ and $p_0$ are the density and pressure at $x_3 = 0$.

**7.42.** Show that if body forces are conservative so that $b_i = -\Omega_{,i}$, the Navier-Stokes-Duhem equations for the irrotational motion of a barotropic fluid may be integrated to yield $-\rho(\partial\phi/\partial t + (\nabla\phi)^2/2) + \rho\Omega + P + (\lambda^* + 2\mu^*)\nabla^2\phi = f(t)$. (See Problem 7.15.)

**7.43.** Show that the velocity and vorticity for an inviscid flow having conservative body forces and constant density satisfy the relation $\dot{q}_i - q_j v_{i,j} = 0$. For steady flow of the same fluid, show that $v_j q_{i,j} = q_j v_{i,j}$.

**7.44.** For a barotropic fluid having $\rho = \rho(p)$ and $P(p)$ defined by (*7.29*), show that $\text{grad}\,P = \text{grad}\,p/\rho$.

**7.45.** Show that the Bernoulli equation (*7.39*) for steady motion of an ideal gas takes the form (*a*) $\Omega + p \ln(p/\rho) + v^2/2 =$ constant, for isothermal flow, (*b*) $\Omega + (k/k-1)(p/\rho) + v^2/2 =$ constant, for isentropic flow.

**7.46.** Show that the velocity field $v_1 = -2x_1x_2x_3/r^4$, $v_2 = (x_1^2 - x_2^2)x_3/r^4$, $v_3 = x_2/r^2$ where $r^2 = x_1^2 + x_2^2 + x_3^2$ is a possible flow for an incompressible fluid. Is the motion irrotational? *Ans.* Yes

**7.47.** If the velocity potential $\Phi(z) = \phi + i\psi$ is an analytic function of the complex variable $z = x_1 + ix_2 = re^{i\theta}$ show that in polar coordinates $\dfrac{\partial\phi}{\partial r} = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta}$ and $\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta} = -\dfrac{\partial\psi}{\partial r}$.

**7.48.** If body forces are zero, show that for irrotational potential flow $\psi_{,ii} = \nu^* = \mu^*/\rho$ is the kinematic viscosity.

# Chapter 8

## Plasticity

### 8.1 BASIC CONCEPTS AND DEFINITIONS

Elastic deformations, which were considered in Chapter 6, are characterized by complete recovery to the undeformed configuration upon removal of the applied loads. Also, elastic deformations depend solely upon the stress magnitude and not upon the straining or loading history. Any deformational response of a continuum to applied loads, or to environmental changes, that does not obey the constitutive laws of classical elasticity may be spoken of as an *inelastic deformation*. In particular, irreversible deformations which result from the mechanism of *slip*, or from dislocations at the atomic level, and which thereby lead to permanent dimensional changes are known as *plastic deformations*. Such deformations occur only at stress intensities above a certain threshold value known as the *elastic limit*, or *yield stress*, which is denoted here by $\sigma_Y$.

In the *theory of plasticity*, the primary concerns are with the mathematical formulation of stress-strain relationships suitable for the phenomenological description of plastic deformations, and with the establishment of appropriate yield criteria for predicting the onset of plastic behavior. By contrast, the study of plastic deformation from the microscopic point of view resides in the realm of solid state physics.

The phrase *plastic flow* is used extensively in plasticity to designate an on-going plastic deformation. However, unlike a fluid flow, such a continuing plastic flow may be related to the amount of deformation as well as the rate of deformation. Indeed, a solid in the "plastic" state can sustain shear stresses even when at rest.

Many of the basic concepts of plasticity may be introduced in an elementary way by consideration of the stress-strain diagram for a simple one-dimensional tension (or compression) test of some hypothetical material as shown by Fig. 8-1. In this plot, $\sigma$ is the nominal stress (force/original area), whereas the strain $\epsilon$ may represent either the *conventional* (*engineering*) *strain* defined here by

$$e = (L - L_0)/L_0 \tag{8.1}$$

where $L$ is the current specimen length and $L_0$ the original length, or the *natural* (*logarithmic*) *strain* defined by

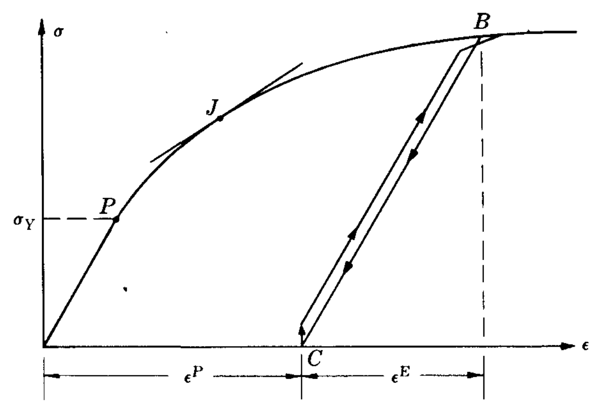



Fig. 8-1

$$\epsilon = \ln(L/L_0) = \ln(1+e) = e - e^2/2 + O(e^3) \tag{8.2}$$

For small strains, these two measures of strain are very nearly equal as seen by (*8.2*) and it is often permissible to neglect the difference.

The yield point $P$, corresponding to the yield stress $\sigma_Y$, separates the stress-strain curve of Fig. 8-1 into an *elastic range* and a *plastic range.* Unfortunately, the yield point is not always well-defined. It is sometimes taken at the *proportional limit,* which lies at the upper end of the linear portion of the curve. It may also be chosen as the point $J$, known as *Johnson's apparent elastic limit,* and defined as that point where the slope of the curve attains 50% of its initial value. Various offset methods are also used to define the yield point, one such being the stress value at 0.2 per cent permanent strain.

In the initial elastic range, which may be linear or nonlinear, an increase in load causes the stress-strain-state-point to move upward along the curve, and a decrease in load, or unloading causes the point to move downward along the same path. Thus a one-to-one stress-strain relationship exists in the elastic range.

In the plastic range, however, unloading from a point such as $B$ in Fig. 8-1 results in the state point following the path $BC$ which is essentially parallel with the linear elastic portion of the curve. At $C$, where the stress reaches zero, the permanent plastic strain $\epsilon^P$ remains. The recoverable elastic strain from $B$ is labeled $\epsilon^E$ in Fig. 8-1. A reloading from $C$ back to $B$ would follow very closely the path $BC$ but with a rounding at $B$, and with a small *hysteresis loop* resulting from the energy loss in the unloading-reloading cycle. Upon a return to $B$ a load increase is required to cause further deformation, a condition referred to as *work hardening,* or *strain hardening.* It is clear therefore that in the plastic range the stress depends upon the entire loading, or strain history of the material.

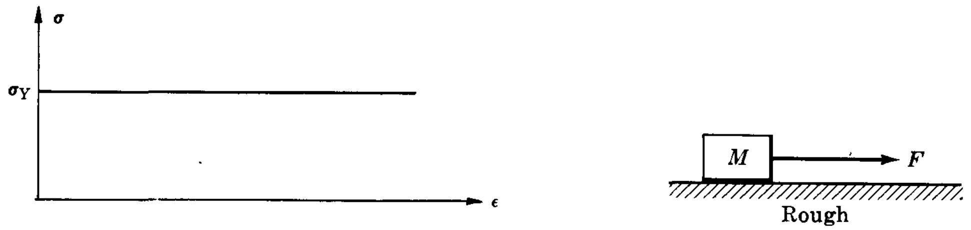
Although it is recognized that temperature will have a definite influence upon the plastic behavior of a real material, it is customary in much of plasticity to assume isothermal conditions and consider temperature as a parameter. Likewise, it is common practice in traditional plasticity to neglect any effect that rate of loading would have upon the stress-strain curve. Accordingly, plastic deformations are assumed to be time-independent and separate from such phenomena as creep and relaxation.

## 8.2 IDEALIZED PLASTIC BEHAVIOR

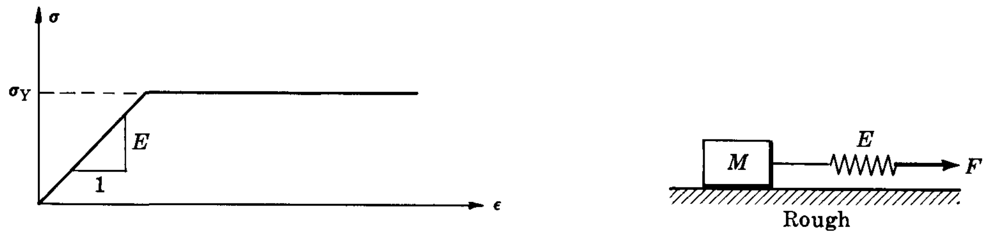
Much of the three-dimensional theory for analyzing plastic behavior may be looked upon as a generalization of certain idealizations of the one-dimensional stress-strain curve of Fig. 8-1. The four most commonly used of these idealized stress-strain diagrams are shown in Fig. 8-2 below, along with a simple mechanical model of each. In the models the displacement of the mass depicts the plastic deformation, and the force $F$ plays the role of stress.

In Fig. 8-2(*a*), elastic response and work-hardening are missing entirely, whereas in (*b*), elastic response prior to yield is included but work-hardening is not. In the absence of work-hardening the plastic response is termed *perfectly plastic.* Representations (*a*) and (*b*) are especially useful in studying *contained plastic deformation,* where large deformations are prohibited. In Fig. 8-2(*c*), elastic response is omitted and the work-hardening is assumed to be *linear.* This representation, as well as (*a*), has been used extensively in analyzing *uncontained plastic flow.*

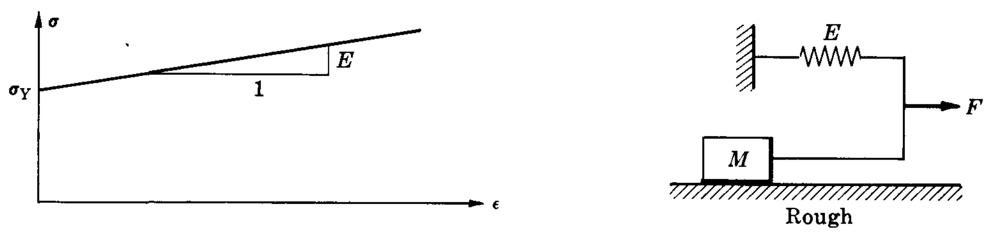
The stress-strain curves of Fig. 8-2 appear in the context of tension curves. The compression curve for a previously unworked specimen (no history of plastic deformation) is taken as the reflection with respect to the origin of the tension curve. However, if a stress reversal (tension to compression, or vice versa) is carried out with a real material that has been work-hardened, a definite lowering of the yield stress is observed in the second type of loading. This phenomenon is known as the *Bauschinger effect,* and will be neglected in this book.



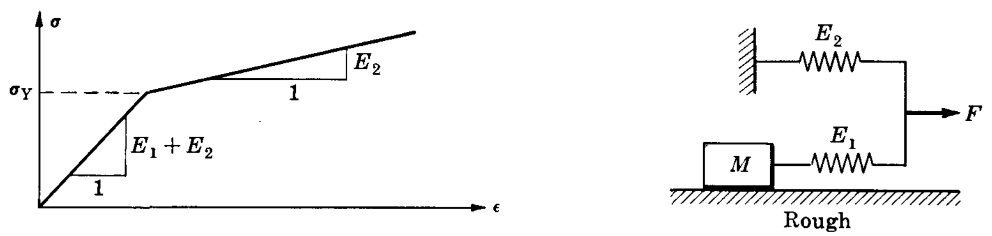

(*a*) Rigid—Perfectly Plastic




(*b*) Elastic—Perfectly Plastic




(*c*) Rigid—Linear Work Hardening




(*d*) Elastic—Linear Work Hardening

Fig. 8-2

## 8.3 YIELD CONDITIONS. TRESCA AND VON MISES CRITERIA

A *yield condition* is essentially a generalization to a three-dimensional state of stress of the yield stress concept in one dimensional loading. Briefly, the yield condition is a mathematical relationship among the stress components at a point that must be satisfied for the onset of plastic behavior at the point. In general, the yield condition may be expressed by the equation

$$f(\sigma_{ij}) = C_Y \tag{8.3}$$

where $C_Y$ is known as the *yield constant*, or as is sometimes done by the equation

$$f_1(\sigma_{ij}) = 0 \tag{8.4}$$

in which $f_1(\sigma_{ij})$ is called the *yield function.*

For an isotropic material the yield condition must be independent of direction and may therefore be expressed as a function of the stress invariants, or alternatively, as a symmetric function of the principal stresses. Thus (*8.3*) may appear as

$$f_2(\sigma_{\rm I}, \sigma_{\rm II}, \sigma_{\rm III}) = C_{\rm Y} \tag{8.5}$$

Furthermore, experiment indicates that yielding is unaffected by moderate hydrostatic stress so that it is possible to present the yield condition as a function of the stress deviator invariants in the form

$$f_3(\mathrm{II}_{\Sigma_D}, \mathrm{III}_{\Sigma_D}) = 0 \tag{8.6}$$

Of the numerous yield conditions which have been proposed, two are reasonably simple mathematically and yet accurate enough to be highly useful for the initial yield of isotropic materials. These are:

(1) *Tresca yield condition* (Maximum Shear Theory)

This condition asserts that yielding occurs when the maximum shear stress reaches the prescribed value $C_{\rm Y}$. Mathematically, the condition is expressed in its simplest form when given in terms of principal stresses. Thus for $\sigma_{\rm I} > \sigma_{\rm II} > \sigma_{\rm III}$, the Tresca yield condition is given from (*2.54b*) as

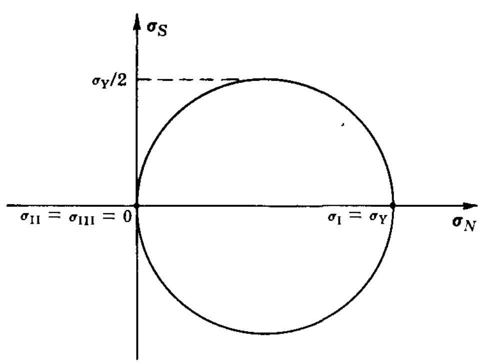
$$\tfrac{1}{2}(\sigma_{\rm I} - \sigma_{\rm III}) = C_{\rm Y} \quad \text{(a constant)} \tag{8.7}$$

To relate the yield constant $C_{\rm Y}$ to the yield stress in simple tension $\sigma_{\rm Y}$, the maximum shear in simple tension at yielding is observed (by the Mohr's circles of Fig. 8-3(*a*), for example) to be $\sigma_{\rm Y}/2$. Therefore when referred to the yield stress in simple tension, Tresca's yield condition becomes

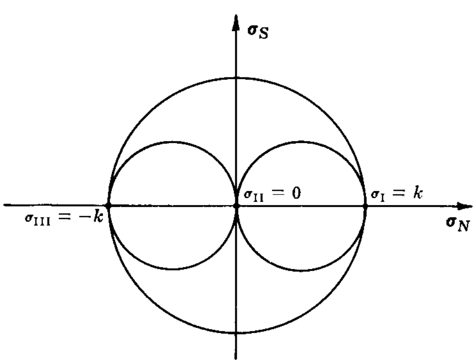
$$\sigma_{\rm I} - \sigma_{\rm III} = \sigma_{\rm Y} \tag{8.8}$$

The yield point for a state of stress that is so-called *pure shear* may also be used as a reference stress in establishing the yield constant $C_{\rm Y}$. Thus if the pure shear yield point value is $k$, the yield constant $C_{\rm Y}$ equals $k$ (again the Mohr's circles clearly show this result, as in Fig. 8-3(*b*), and the Tresca yield criterion is written in the form

$$\sigma_{\rm I} - \sigma_{\rm III} = 2k \tag{8.9}$$




(*a*) Simple Tension




(*b*) Pure Shear

**Fig. 8-3**

(2) *von Mises yield condition* (Distortion Energy Theory)

This condition asserts that yielding occurs when the second deviator stress invariant attains a specified value. Mathematically, the von Mises yield condition states

$$-\mathrm{II}_{\Sigma_D} = C_Y \qquad (8.10)$$

which is usually written in terms of the principal stresses as

$$(\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})^2 + (\sigma_{\mathrm{II}} - \sigma_{\mathrm{III}})^2 + (\sigma_{\mathrm{III}} - \sigma_{\mathrm{I}})^2 = 6C_Y \qquad (8.11)$$

With reference to the yield stress in simple tension, it is easily shown that (*8.11*) becomes

$$(\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})^2 + (\sigma_{\mathrm{II}} - \sigma_{\mathrm{III}})^2 + (\sigma_{\mathrm{III}} - \sigma_{\mathrm{I}})^2 = 2\sigma_Y^2 \qquad (8.12)$$

Also, with respect to the pure shear yield value $k$, von Mises condition (*8.11*) appears in the form

$$(\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}})^2 + (\sigma_{\mathrm{II}} - \sigma_{\mathrm{III}})^2 + (\sigma_{\mathrm{III}} - \sigma_{\mathrm{I}})^2 = 6k^2 \qquad (8.13)$$

There are several variations for presenting (*8.12*) and (*8.13*) when stress components other than the principal stresses are employed.

## 8.4 STRESS SPACE. THE Π-PLANE. YIELD SURFACE

A stress space is established by using stress magnitude as the measure of distance along the coordinate axes. In the Haigh-Westergaard stress space of Fig. 8-4 the coordinate axes are associated with the principal stresses. Every point in this space corresponds to a state of stress, and the position vector of any such point $P(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}, \sigma_{\mathrm{III}})$ may be resolved into a component $OA$ along the line $OZ$, which makes equal angles with the coordinate axes, and a component $OB$ in the plane (known as the Π-plane) which is perpendicular to $OZ$ and passes through the origin. The component along $OZ$, for which $\sigma_{\mathrm{I}} = \sigma_{\mathrm{II}} = \sigma_{\mathrm{III}}$, represents hydrostatic stress, so that the component in the Π-plane represents the deviator portion of the stress state. It is easily shown that the equation of the Π-plane is given by

$$\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}} + \sigma_{\mathrm{III}} = 0 \qquad (8.14)$$

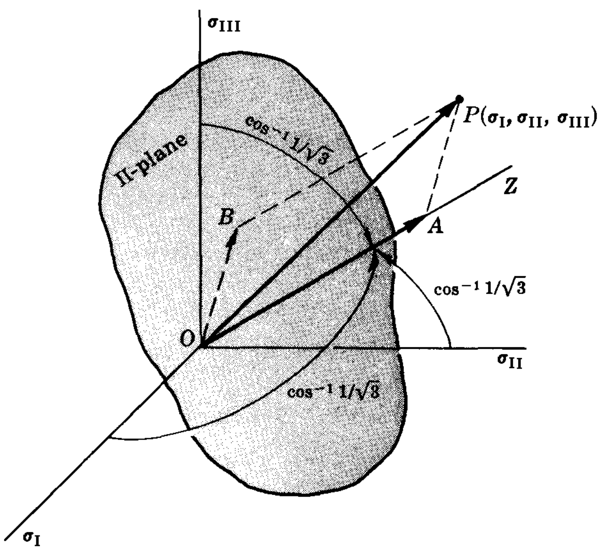



Fig. 8-4

In stress space, the yield condition (*8.5*), $f_2(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}, \sigma_{\mathrm{III}}) = C_Y$, defines a surface, the so-called *yield surface*. Since the yield conditions are independent of hydrostatic stress, such yield surfaces are general cylinders having their generators parallel to $OZ$. Stress points that lie inside the cylindrical yield surface represent elastic stress states, those which lie on the yield surface represent incipient plastic stress states. The intersection of the yield surface with the Π-plane is called the *yield curve*.

In a true view of the Π-plane, looking along $OZ$ toward the origin $O$, the principal stress axes appear symmetrically placed 120° apart as shown in Fig. 8-5(*a*) below. The yield curves for the Tresca and von Mises yield conditions appear in the Π-plane as shown in Fig. 8-5(*b*) and (*c*) below. In Fig. 8-5(*b*), these curves are drawn with reference to (*8.7*) and (*8.11*), using the yield stress in simple tension as the basis. For this situation, the von Mises circle of radius $\sqrt{2/3}\,\sigma_Y$ is seen to circumscribe the regular Tresca hexagon. In Fig. 8-5(*c*), the two yield curves are based upon the yield stress $k$ in pure shear. Here the von Mises circle is inscribed in the Tresca hexagon.

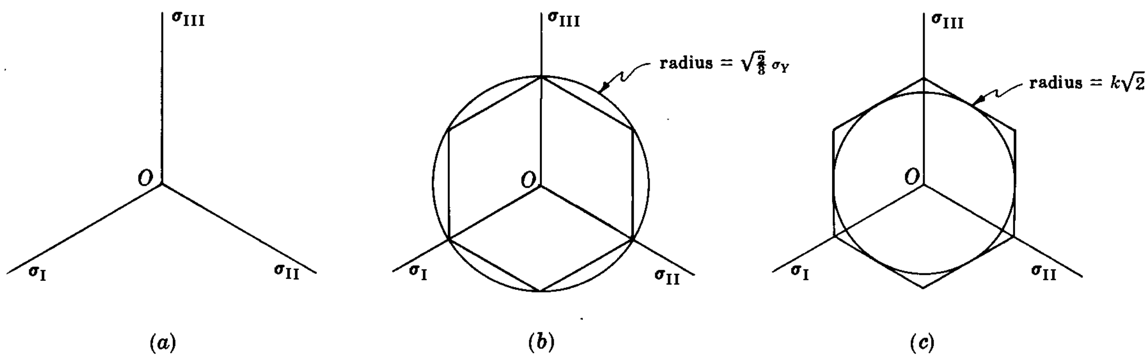



**Fig. 8-5**

The location in the Π-plane of the projection of an arbitrary stress point $P(\sigma_I, \sigma_{II}, \sigma_{III})$ is straightforward since each of the stress space axes makes $\cos^{-1}\sqrt{2/3}$ with the Π-plane. Thus the projected deviatoric components are $(\sqrt{2/3}\,\sigma_I, \sqrt{2/3}\,\sigma_{II}, \sqrt{2/3}\,\sigma_{III})$. The inverse problem of determining the stress components for an arbitrary point in the Π-plane is not unique since the hydrostatic stress component may have any value.

## 8.5 POST-YIELD BEHAVIOR. ISOTROPIC AND KINEMATIC HARDENING

Continued loading after initial yield is reached leads to plastic deformation which may be accompanied by changes in the yield surface. For an assumed *perfectly plastic* material the yield surface does not change during plastic deformation and the initial yield condition remains valid. This corresponds to the one-dimensional perfectly plastic case depicted by Fig. 8-2(*a*). For a *strain hardening* material, however, plastic deformation is generally accompanied by changes in the yield surface. To account for such changes it is necessary that the yield function $f_1(\sigma_{ij})$ of (*8.4*) be generalized to define subsequent yield surfaces beyond the initial one. A generalization is effected by introduction of the *loading function*

$$f_1^*(\sigma_{ij}, \epsilon_{ij}^P, K) = 0 \tag{8.15}$$

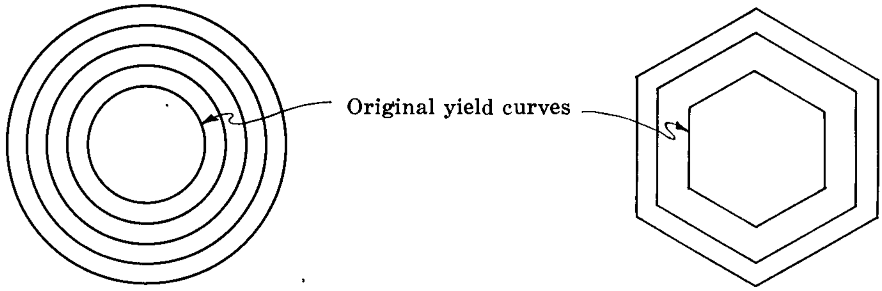
which depends not only upon the stresses, but also upon the plastic strains $\epsilon_{ij}^P$ and the work-hardening characteristics represented by the parameter $K$. Equation (*8.15*) defines a loading surface in the sense that $f_1^* = 0$ is the yield surface, $f_1^* < 0$ is a surface in the (elastic) region inside the yield surface and $f_1^* > 0$, being outside the yield surface, has no meaning.

Differentiating (*8.15*) by the chain rule of calculus,

$$df_1^* = \frac{\partial f_1^*}{\partial \sigma_{ij}}\,d\sigma_{ij} + \frac{\partial f_1^*}{\partial \epsilon_{ij}^P}\,d\epsilon_{ij}^P + \frac{\partial f_1^*}{\partial K}\,dK \tag{8.16}$$

Thus with $f_1^* = 0$ and $(\partial f_1^*/\partial\sigma_{ij})\,d\sigma_{ij} < 0$, *unloading* is said to occur; with $f_1^* = 0$ and $(\partial f_1^*/\partial\sigma_{ij})\,d\sigma_{ij} = 0$, *neutral loading* occurs; and with $f_1^* = 0$ and $(\partial f_1^*/\partial\sigma_{ij})\,d\sigma_{ij} > 0$, *loading* occurs. The manner in which the plastic strains $\epsilon_{ij}^P$ enter into the function (*8.15*) when loading occurs is defined by the *hardening rules*, two especially simple cases of which are described in what follows.

The assumption of *isotropic hardening* under loading conditions postulates that the yield surface simply increases in size and maintains its original shape. Thus in the Π-plane the yield curves for von Mises and Tresca conditions are the concentric circles and regular hexagons shown in Fig. 8-6 below.




(a) Mises Circles (b) Tresca Hexagons

Fig. 8-6

In *kinematic hardening,* the initial yield surface is translated to a new location in stress space without change in size or shape. Thus (8.4) defining an initial yield surface is replaced by

$$f_1(\sigma_{ij} - \alpha_{ij}) = 0 \qquad (8.17)$$

where the $\alpha_{ij}$ are coordinates of the center of the new yield surface. If *linear hardening* is assumed,

$$\dot{\alpha}_{ij} = c\dot{\epsilon}^P_{ij} \qquad (8.18)$$

where $c$ is a constant. In a one-dimensional case, the Tresca yield curve would be translated as shown in Fig. 8-7.

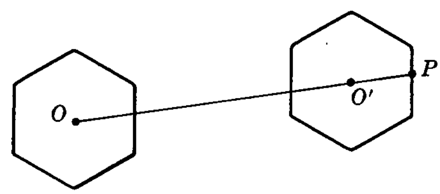



Fig. 8-7

## 8.6 PLASTIC STRESS-STRAIN EQUATIONS. PLASTIC POTENTIAL THEORY

Once plastic deformation is initiated, the constitutive equations of elasticity are no longer valid. Because plastic strains depend upon the entire loading history of the material, plastic stress-strain relations very often are given in terms of strain increments – the so-called *incremental theories.* By neglecting the elastic portion and by assuming that the principal axes of strain increment coincide with the principal stress axes, the *Levy-Mises* equations relate the total strain increments to the deviatoric stress components through the equations

$$d\epsilon_{ij} = s_{ij}\, d\lambda \qquad (8.19)$$

Here the proportionality factor $d\lambda$ appears in differential form to emphasize that incremental strains are being related to finite stress components. The factor $d\lambda$ may change during loading and is therefore a scalar multiplier and not a fixed constant. Equations (8.19) represent the *flow rule* for a rigid-perfectly plastic material.

If the strain increment is split into elastic and plastic portions according to

$$d\epsilon_{ij} = d\epsilon^E_{ij} + d\epsilon^P_{ij} \qquad (8.20)$$

and the plastic strain increments related to the stress deviator components by

$$d\epsilon^P_{ij} = s_{ij}\, d\lambda \qquad (8.21)$$

the resulting equations are known as the *Prandtl-Reuss* equations. Equations (8.21) represent the flow rule for an elastic-perfectly plastic material. They provide a relationship between the plastic strain increments and the current stress deviators but do not specify the strain increment magnitudes.

The name *plastic potential function* is given to that function of the stress components $g(\sigma_{ij})$ for which

$$d\epsilon_{ij}^P = \frac{\partial g}{\partial \sigma_{ij}} d\lambda \tag{8.22}$$

For a so-called stable plastic material such a function exists and is identical to the yield function. Moreover when the yield function $f_1(\sigma_{ij}) = \text{II}_{\Sigma_D}$, (*8.22*) produces the Prandtl-Reuss equations (*8.21*).

## 8.7 EQUIVALENT STRESS. EQUIVALENT PLASTIC STRAIN INCREMENT

With regard to the mathematical formulation of strain hardening rules, it is useful to define the *equivalent* or *effective stress* $\sigma_{EQ}$ as

$$\sigma_{EQ} = \frac{1}{\sqrt{2}}\{[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2] + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2)\}^{1/2} \tag{8.23}$$

This expression may be written in compact form as

$$\sigma_{EQ} = \sqrt{3s_{ij}s_{ij}/2} = \sqrt{3\text{II}_{\Sigma_D}} \tag{8.24}$$

In a similar fashion, the *equivalent* or *effective plastic strain increment* $d\epsilon_{EQ}^P$ is defined by

$$d\epsilon_{EQ}^P = \{\tfrac{2}{9}[(d\epsilon_{11}^P - d\epsilon_{22}^P)^2 + (d\epsilon_{22}^P - d\epsilon_{33}^P)^2 + (d\epsilon_{33}^P - d\epsilon_{11}^P)^2] + \tfrac{4}{3}[(d\epsilon_{12}^P)^2 + (d\epsilon_{23}^P)^2 + (d\epsilon_{31}^P)^2]\}^{1/2} \tag{8.25}$$

which may be written compactly in the form

$$d\epsilon_{EQ}^P = \sqrt{\tfrac{2}{3}\, d\epsilon_{ij}^P\, d\epsilon_{ij}^P} \tag{8.26}$$

In terms of the equivalent stress and strain increments defined by (*8.24*) and (*8.25*) respectively, $d\lambda$ of (*8.21*) becomes

$$d\lambda = \frac{3}{2}\frac{d\epsilon_{EQ}^P}{\sigma_{EQ}} \tag{8.27}$$

## 8.8 PLASTIC WORK. STRAIN-HARDENING HYPOTHESES

The rate at which the stresses do work, or the *stress power* as it is called, has been given in (*5.32*) as $\sigma_{ij}D_{ij}$ per unit volume. From (*4.25*), $d\epsilon_{ij} = D_{ij}\,dt$, so that the work increment per unit volume may be written

$$dW = \sigma_{ij}\, d\epsilon_{ij} \tag{8.28}$$

and using (*8.20*) this may be split into

$$dW = \sigma_{ij}(d\epsilon_{ij}^E + d\epsilon_{ij}^P) = dW^E + dW^P \tag{8.29}$$

For a plastically incompressible material, the *plastic work increment* becomes

$$dW^P = \sigma_{ij}\, d\epsilon_{ij}^P = s_{ij}\, d\epsilon_{ij}^P \tag{8.30}$$

Furthermore, if the same material obeys the Prandtl-Reuss equations (*8.21*), the plastic work increment may be expressed as

$$dW^P = \sigma_{EQ}\, d\epsilon_{EQ}^P \tag{8.31}$$

and (*8.21*) rewritten in the form

$$d\epsilon_{ij}^P = \frac{3}{2}\frac{dW^P}{\sigma_{EQ}^2} s_{ij} \tag{8.32}$$

There are two widely considered hypotheses proposed for computing the current yield stress under *isotropic strain hardening* plastic flow. One, known as the *work-hardening hypothesis*, assumes that the current yield surface depends only upon the total plastic work done. Thus with the total plastic work given as the integral

$$W^P = \int \sigma_{ij}\, d\epsilon_{ij}^P \tag{8.33}$$

the yield criterion may be expressed symbolically by the equation

$$f_1(\sigma_{ij}) = F(W^P) \tag{8.34}$$

for which the precise functional form must be determined experimentally. A second hardening hypothesis, known as the *strain-hardening hypothesis*, assumes that the hardening is a function of the amount of plastic strain. In terms of the total equivalent strain

$$\epsilon_{EQ}^P = \int d\epsilon_{EQ}^P \tag{8.35}$$

this hardening rule is expressed symbolically by the equation

$$f_1(\sigma_{ij}) = H(\epsilon_{EQ}^P) \tag{8.36}$$

for which the functional form is determined from a uniaxial stress-strain test of the material. For the Mises yield criterion, the hardening rules (*8.34*) and (*8.36*) may be shown to be equivalent.

## 8.9 TOTAL DEFORMATION THEORY

In contrast to the *incremental theory* of plastic strain as embodied in the stress-strain increment equations (*8.19*) and (*8.21*), the so-called *total deformation theory* of Hencky relates stress and total strain. The equations take the form

$$e_{ij} = (\phi + \tfrac{1}{2}G)s_{ij} \tag{8.37}$$

$$\epsilon_{ii} = (1-2\nu)\sigma_{ii}/E \tag{8.38}$$

In terms of equivalent stress and strain, the parameter $\phi$ may be expressed as

$$\phi = \frac{3}{2}\frac{\epsilon_{EQ}^P}{\sigma_{EQ}} \tag{8.39}$$

where here $\epsilon_{EQ}^P = \sqrt{2\epsilon_{ij}^P\epsilon_{ij}^P/3}$ so that

$$\epsilon_{ij}^P = \frac{3}{2}\frac{\epsilon_{EQ}^P}{\sigma_{EQ}}s_{ij} \tag{8.40}$$

## 8.10 ELASTOPLASTIC PROBLEMS

Situations in which both elastic and plastic strains of approximately the same order exist in a body under load are usually referred to as *elasto-plastic problems*. A number of well-known examples of such problems occur in beam theory, torsion of shafts and thick-walled tubes and spheres subjected to pressure. In general, the governing equations for the elastic region, the plastic region and the elastic-plastic interface are these:

(*a*) Elastic region

1. Equilibrium equations (*2.23*), page 49
2. Stress-strain relations (*6.23*) or (*6.24*), page 143
3. Boundary conditions on stress or displacement
4. Compatibility conditions

(*b*) Plastic region

1. Equilibrium equations (*2.23*), page 49
2. Stress-strain increment relations (*8.21*)
3. Yield condition (*8.8*) or (*8.11*)
4. Boundary conditions on plastic boundary when such exists

(*c*) Elastic-plastic interface

1. Continuity conditions on stress and displacement

## 8.11 ELEMENTARY SLIP LINE THEORY FOR PLANE PLASTIC STRAIN

In unrestricted plastic flow such as occurs in metal-forming processes, it is often possible to neglect elastic strains and consider the material to be rigid-perfectly plastic. If the flow may be further assumed to be a case of plane strain, the resulting velocity field may be studied using *slip line theory.*

Taking the $x_1x_2$ plane as the plane of flow, the stress tensor is given in the form

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \tag{8.41}$$

and since elastic strains are neglected, the plastic strain-rate tensor applicable to the situation is

$$\dot{\epsilon}_{ij} = \begin{pmatrix} \dot{\epsilon}_{11} & \dot{\epsilon}_{12} & 0 \\ \dot{\epsilon}_{12} & \dot{\epsilon}_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{8.42}$$

In (*8.41*) and (*8.42*) the variables are functions of $x_1$ and $x_2$ only, and also

$$\dot{\epsilon}_{ij} = \tfrac{1}{2}(v_{i,j} + v_{i,j}) \tag{8.43}$$

where $v_i$ are the velocity components.

For the assumed plane strain condition, $d\epsilon_{33} = 0$; and so from the Prandtl-Reuss equations (*8.21*), the stress $\sigma_{33}$ is given by

$$\sigma_{33} = \tfrac{1}{2}(\sigma_{11} + \sigma_{22}) \tag{8.44}$$

Adopting the standard slip-line notation $\sigma_{33} = -p$, and $\sqrt{(\sigma_{11} - \sigma_{22})^2/4 + (\sigma_{12})^2} = k$, the principal stress values of (*8.41*) are found to be

$$\begin{aligned} \sigma_{(1)} &= -p + k \\ \sigma_{(2)} &= -p \\ \sigma_{(3)} &= -p - k \end{aligned} \tag{8.45}$$

The principal stress directions are given with respect to the $x_1x_2$ axes as shown in Fig. 8-8, where $\tan 2\theta = 2\sigma_{12}/(\sigma_{11} - \sigma_{22})$.

As was shown in Section 2.11, the maximum shear directions are at 45° with respect to the principal stress directions. In Fig. 8-8, the maximum shear directions are designated as the $\alpha$ and $\beta$ directions. From the

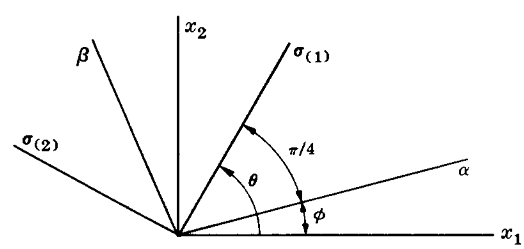


Fig. 8-8

geometry of this diagram, $\theta = \pi/4 + \phi$ so that

$$\tan 2\phi = -\frac{1}{\tan 2\theta} \tag{8.46}$$

and for a given stress field in a plastic flow, two families of curves along the directions of maximum shear at every point may be established. These curves are called *shear lines*, or *slip lines*.

For a small curvilinear element bounded by the two pairs of slip lines shown in Fig. 8-9,

$$\begin{aligned} \sigma_{11} &= -p - k\sin 2\phi \\ \sigma_{22} &= -p + k\sin 2\phi \\ \sigma_{12} &= k\cos 2\phi \end{aligned} \tag{8.47}$$

and from the equilibrium equations it may be shown that

$$\begin{aligned} p + 2k\phi &= C_1 \quad \text{a constant along an } \alpha \text{ line} \\ p - 2k\phi &= C_2 \quad \text{a constant along a } \beta \text{ line} \end{aligned} \tag{8.48}$$

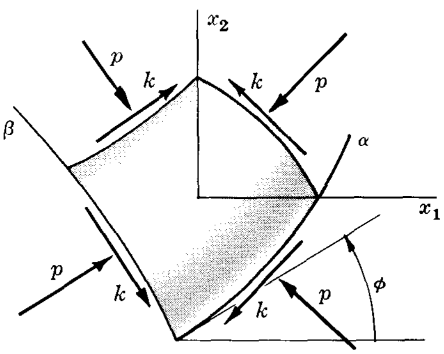



Fig. 8-9

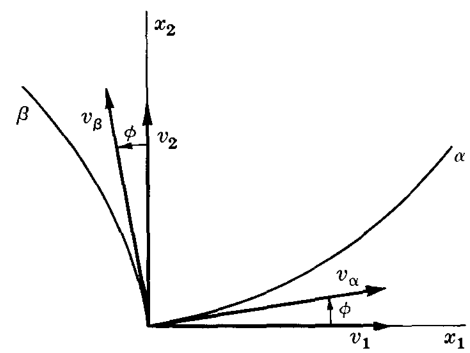



Fig. 8-10

With respect to the velocity components, Fig. 8-10 shows that relative to the $\alpha$ and $\beta$ lines,

$$\begin{aligned} v_1 &= v_\alpha \cos\phi - v_\beta \sin\phi \\ v_2 &= v_\alpha \sin\phi + v_\beta \cos\phi \end{aligned} \tag{8.49}$$

For an isotropic material, the principal axes of stress and plastic strain-rate coincide. Therefore if $x_1$ and $x_2$ are slip-line directions, $\dot{\epsilon}_{11}$ and $\dot{\epsilon}_{22}$ are zero along the slip-lines so that

$$\left\{\frac{\partial}{\partial x_1}(v_\alpha \cos\phi - v_\beta \sin\phi)\right\}_{\phi=0} = 0 \tag{8.50}$$

$$\left\{\frac{\partial}{\partial x_2}(v_\alpha \sin\beta + v_\beta \cos\phi)\right\}_{\phi=0} = 0 \tag{8.51}$$

These equations lead to the relationships

$$dv_1 - v_2\,d\phi = 0 \quad \text{on } \alpha \text{ lines} \tag{8.52}$$

$$dv_2 + v_1\,d\phi = 0 \quad \text{on } \beta \text{ lines} \tag{8.53}$$

Finally, for statically determinate problems, the slip line field may be found from (8.48), and using this slip line field, the velocity field may be determined from (8.52) and (8.53).

# Solved Problems

## BASIC CONCEPTS. YIELD PHENOMENA (Sec. 8.1-8.4)

**8.1.** Making use of the definitions (*8.1*) and (*8.2*), derive the relationship between natural and engineering strain. How are the strain increments of these quantities related?

From (*8.1*), $L/L_0 = e + 1$ and so (*8.2*) becomes $\epsilon = \ln(e+1)$. Differentiating this equation, $d\epsilon/de = 1/(e+1) = L_0/L$ since $dL = L\,d\epsilon = L_0\,de$.

**8.2.** Under a load $P$ in a one-dimensional test the true stress is $\sigma = P/A$ while the engineering stress is $S = P/A_0$ where $A_0$ is the original area and $A$ is the current area. For a constant volume plastic deformation ($A_0L_0 = AL$), determine the condition for maximum load.

Here $S = P/A_0 = (P/A)(A/A_0) = \sigma(L_0/L) = \sigma/(1+e)$, and on an $S$-$e$ plot the maximum load occurs where the slope $dS/de = 0$. Differentiation gives $dS/de = (d\sigma/d\epsilon - \sigma)/(1+e)^2$ and this is zero when $d\sigma/d\epsilon = \sigma$. From Problem 8.1, this condition may be expressed by $d\sigma/de = \sigma/(1+e)$.

**8.3.** As a measure of the influence of the intermediate principal stress in yielding the Lode parameter, $\mu = (2\sigma_{II} - \sigma_I - \sigma_{III})/(\sigma_I - \sigma_{III})$ is often used. Show that in terms of the principal stress deviators this becomes $\mu = 3s_{II}/(s_I - s_{III})$.

From (*2.71*), $\sigma_I = s_I + \sigma_M$, etc., with $\sigma_M = \sigma_{ii}/3$. Thus

$$\begin{aligned}\mu &= [2(s_{II}+\sigma_M) - (s_I+\sigma_M) - (s_{III}+\sigma_M)]/[(s_I+\sigma_M) - (s_{III}+\sigma_M)] \\ &= [3s_{II} - (s_I + s_{II} + s_{III})]/(s_I - s_{III})\end{aligned}$$

But $s_I + s_{II} + s_{III} = I_{\Sigma_D} \equiv 0$ and so $\mu = 3s_{II}/(s_I - s_{III})$.

**8.4.** For the state of stress $\sigma_{11} = \sigma$, $\sigma_{22} = \sigma_{33} = 0$, $\sigma_{12} = \tau$, $\sigma_{23} = \sigma_{13} = 0$ produced in a tension-torsion test of a thin-walled tube, derive the yield curves in the $\sigma$-$\tau$ plane for the Tresca and von Mises conditions if the yield stress in simple tension is $\sigma_Y$.

For the given state of stress the principal stress values are $\sigma_I = (\sigma + \sqrt{4\tau^2+\sigma^2})/2$, $\sigma_{II} = 0$, $\sigma_{III} = (\sigma - \sqrt{4\tau^2+\sigma^2})/2$ as shown by the Mohr's diagram in Fig. 8-11. Thus from (*8.8*) the Tresca yield curve is $\sqrt{4\tau^2+\sigma^2} = \sigma_Y$, or $\sigma^2 + 4\tau^2 = \sigma_Y^2$, an ellipse in the $\sigma$-$\tau$ plane. Likewise from (*8.12*) the Mises yield curve is the ellipse $\sigma^2 + 3\tau^2 = \sigma_Y^2$. The Tresca and Mises yield ellipses for this case are compared in the plot shown in Fig. 8-12.

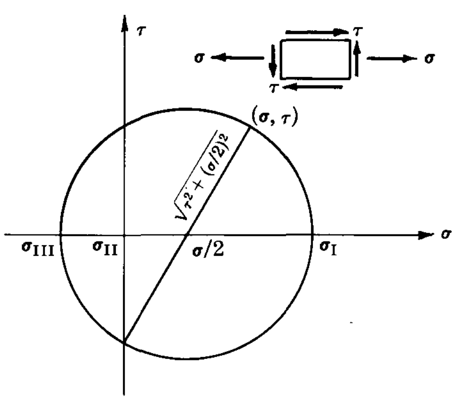



Fig. 8-11

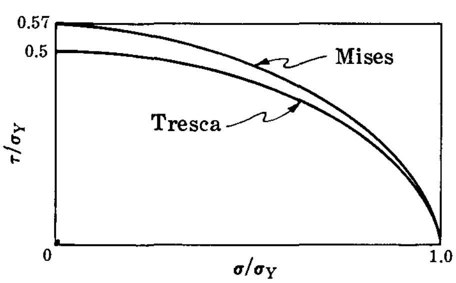



Fig. 8-12

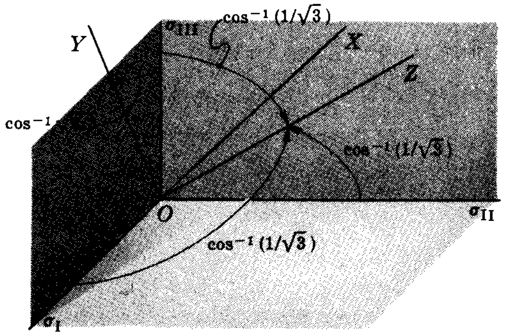
**8.5.** Convert the von Mises yield condition (*8.10*) to its principal stress form as given in (*8.11*).

From (*2.72*), $-\text{II}_{\Sigma_D} = -(s_\text{I}s_\text{II} + s_\text{II}s_\text{III} + s_\text{III}s_\text{I})$; and by (*2.71*), $s_\text{I} = \sigma_\text{I} - \sigma_M$, etc., where $\sigma_M = (\sigma_\text{I} + \sigma_\text{II} + \sigma_\text{III})/3$. Hence

$$-\text{II}_{\Sigma_D} = -(\sigma_\text{I}\sigma_\text{II} + \sigma_\text{II}\sigma_\text{III} + \sigma_\text{III}\sigma_\text{I}) + (\sigma_\text{I} + \sigma_\text{II} + \sigma_\text{III})^2/3$$
$$= 2(\sigma_\text{I}^2 + \sigma_\text{II}^2 + \sigma_\text{III}^2 - \sigma_\text{I}\sigma_\text{II} - \sigma_\text{II}\sigma_\text{III} - \sigma_\text{III}\sigma_\text{I})/6$$

Thus
$$(\sigma_\text{I} - \sigma_\text{II})^2 + (\sigma_\text{II} - \sigma_\text{III})^2 + (\sigma_\text{III} - \sigma_\text{I})^2 = 6C_Y$$

**8.6.** With the rectangular coordinate system $OXYZ$ oriented so that the $XY$ plane coincides with the Π-plane and the $\sigma_\text{III}$ axis lies in the $YOZ$ plane (see Fig. 8-13 and 8-4), show that the Mises yield surface intersects the Π-plane in the Mises circle of Fig. 8-5(*b*).




| | $\sigma_\text{I}$ | $\sigma_\text{II}$ | $\sigma_\text{III}$ |
|---|---|---|---|
| $X$ | $-1/\sqrt{2}$ | $1/\sqrt{2}$ | 0 |
| $Y$ | $-1/\sqrt{6}$ | $-1/\sqrt{6}$ | $2/\sqrt{6}$ |
| $Z$ | $1/\sqrt{3}$ | $1/\sqrt{3}$ | $1/\sqrt{3}$ |

Fig. 8-13

The table of transformation coefficients between the two sets of axes is readily determined to be as shown above. Therefore

$$\sigma_\text{I} = -X/\sqrt{2} - Y/\sqrt{6} + Z/\sqrt{3}, \qquad \sigma_\text{II} = X/\sqrt{2} - Y/\sqrt{6} + Z/\sqrt{3}, \qquad \sigma_\text{III} = 2Y/\sqrt{6} + Z/\sqrt{3}$$

and (*8.12*) becomes
$$(-\sqrt{2}\,X)^2 + (X/\sqrt{2} - 3Y/\sqrt{6})^2 + (X/\sqrt{2} + 3Y/\sqrt{6})^2 = 2\sigma_Y^2$$

which simplifies to the Mises yield circle $3X^2 + 3Y^2 = 2\sigma_Y^2$ of Fig. 8-5(*b*).

**8.7.** Using the transformation equations of Problem 8.6, show that (*8.14*), $\sigma_\text{I} + \sigma_\text{II} + \sigma_\text{III} = 0$, is the equation of the Π-plane.

Substituting into (*8.14*) the σ's of Problem 8.6, $\sigma_\text{I} + \sigma_\text{II} + \sigma_\text{III} = \sqrt{3}\,Z = 0$, or $Z = 0$ which is the $XY$ plane (Π-plane).

**8.8.** For a biaxial state of stress with $\sigma_\text{II} = 0$, determine the yield loci for the Mises and Tresca conditions and compare them by a plot in the two-dimensional $\sigma_\text{I}/\sigma_Y$ vs. $\sigma_\text{III}/\sigma_Y$ space.

From (*8.12*) with $\sigma_\text{II} = 0$, the Mises yield condition becomes
$$\sigma_\text{I}^2 - \sigma_\text{I}\sigma_\text{III} + \sigma_\text{III}^2 = \sigma_Y^2$$

which is the ellipse
$$(\sigma_\text{I}/\sigma_Y)^2 - (\sigma_\text{I}\sigma_\text{III}/\sigma_Y^2) + (\sigma_\text{III}/\sigma_Y)^2 = 1$$

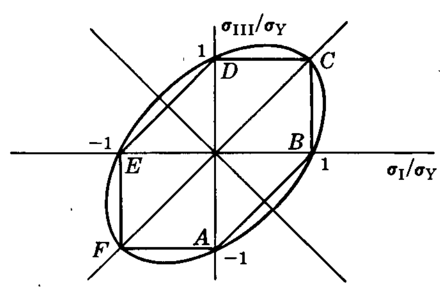



Fig. 8-14

with axes at 45° in the plot. Likewise, from (*8.8*) and the companion equations $\sigma_{III} - \sigma_{II} = \sigma_Y$, $\sigma_{II} - \sigma_I = \sigma_Y$, the Tresca yield condition results in the line segments $AB$ and $ED$ with equations $(\sigma_I/\sigma_Y) - (\sigma_{III}/\sigma_Y) = \pm 1$, $DC$ and $FA$ with equations $\sigma_{III}/\sigma_Y = \pm 1$, and $BC$ and $EF$ with equations $\sigma_I/\sigma_Y = \mp 1$, respectively.

**8.9.** The von Mises yield condition is referred to in Section **8.3** as the Distortion Energy Theory. Show that if the distortion energy per unit volume $u^*_{(D)}$ is set equal to the yield constant $C_Y$ the result is the Mises criterion as given by (*8.12*).

From Problem 6.26, $u^*_{(D)}$ is given in terms of the principal stresses by

$$u^*_{(D)} = [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]/12G$$

and for a uniaxial yield situation where $\sigma_1 = \sigma_Y$, $\sigma_{II} = \sigma_{III} = 0$, $u^*_{(D)} = \sigma_Y^2/6G$. Thus $C_Y = \sigma_Y^2/6G$ and, as before, the Mises yield condition is expressed by (*8.12*).

## PLASTIC DEFORMATION. STRAIN-HARDENING (Sec. 8.4-8.8)

**8.10.** Show that the Prandtl-Reuss equations (*8.21*) imply that principal axes of plastic strain increments coincide with principal stress axes and express the equations in terms of the principal stresses.

From the form of (*8.21*), when referred to a coordinate system in which the shear stresses are zero, the plastic shear strain increments are seen to be zero also. In the principal axes system, (*8.21*) becomes $d\epsilon_I^P/s_I = d\epsilon_{II}^P/s_{II} = d\epsilon_{III}^P/s_{III} = d\lambda$. Thus $d\epsilon_I^P = (\sigma_I - \sigma_M)\,d\lambda$, $d\epsilon_{II}^P = (\sigma_{II} - \sigma_M)\,d\lambda$, etc., and by subtracting,

$$\frac{d\epsilon_I^P - d\epsilon_{II}^P}{\sigma_I - \sigma_{II}} = \frac{d\epsilon_{II}^P - d\epsilon_{III}^P}{\sigma_{II} - \sigma_{III}} = \frac{d\epsilon_{III}^P - d\epsilon_I^P}{\sigma_{III} - \sigma_I} = d\lambda$$

**8.11.** For the case of plastic plane strain with $\epsilon_{33} = 0$, $d\epsilon_{33} = 0$ and $\sigma_{22} = 0$, show that the Levy-Mises equations (*8.19*) lead to the conclusion that the Tresca and Mises yield conditions (when related to pure shear yield stress $k$) are identical.

Here (*8.19*) becomes $d\epsilon_{11} = (2\sigma_{11} - \sigma_{33})\,d\lambda/3$, $d\epsilon_{22} = -(\sigma_{11} + \sigma_{33})\,d\lambda/3$, $0 = 2\sigma_{33} - \sigma_{11}$. Thus in the absence of shear stresses, $\sigma_I = \sigma_{11}$, $\sigma_{II} = \sigma_{33} = \sigma_{11}/2$, $\sigma_{III} = 0 = \sigma_{22}$. Then from (*8.9*) the Tresca yield condition is $\sigma_I - \sigma_{III} = \sigma_{11} = 2k$. Also, from (*8.13*) for this case, Mises condition becomes $(\sigma_{11}/2)^2 + (-\sigma_{11}/2)^2 + (-\sigma_{11})^2 = 6k^2$ or $\sigma_{11}^2 = 4k^2$ and $\sigma_{11} = 2k$.

**8.12.** Show that the Prandtl-Reuss equations imply equality of the Lode variable $\mu$ (see Problem 8.3) and $\nu = (2d\epsilon_{II}^P - d\epsilon_I^P - d\epsilon_{III}^P)/(d\epsilon_I^P - d\epsilon_{III}^P)$.

From equations (*8.21*),

$$\begin{aligned} \nu &= (2s_{II} - s_I - s_{III})\,d\lambda/(s_I - s_{III})\,d\lambda \\ &= (2(\sigma_{II} - \sigma_M) - (\sigma_I - \sigma_M) - (\sigma_{III} - \sigma_M))/((\sigma_I - \sigma_M) - (\sigma_{III} - \sigma_M)) \\ &= (2\sigma_{II} - \sigma_I - \sigma_{III})/(\sigma_I - \sigma_{III}) = \mu \end{aligned}$$

**8.13.** Writing $II_{\Sigma_D} = s_{ij}s_{ij}/2$, show that $\partial II_{\Sigma_D}/\partial\sigma_{ij} = s_{ij}$.

Here $\partial II_{\Sigma_D}/\partial\sigma_{pq} = (\partial s_{ij}/\partial\sigma_{pq})s_{ij}$ where $\partial s_{ij}/\partial\sigma_{pq} = \partial(\sigma_{ij} - \delta_{ij}\sigma_{kk}/3)/\partial\sigma_{pq} = \delta_{ip}\delta_{jq} - \delta_{ij}\delta_{pq}/3$. Thus $\partial II_{\Sigma_D}/\partial\sigma_{pq} = (\delta_{ip}\delta_{jq} - \delta_{ij}\delta_{pq}/3)s_{ij} = s_{pq}$ since $s_{ii} = I_{\Sigma_D} = 0$.

**8.14.** Show that when the plastic potential function $g(\sigma_{ij}) = \text{II}_{\Sigma_D}$, the plastic potential equations (*8.22*) become the **Prandtl-Reuss equations.**

The proof follows directly from the result of Problem 8.13, since $\partial g/\partial \sigma_{ij} = s_{ij}$ in this case and (*8.22*) reduce to (*8.21*).

**8.15.** Expand (*8.24*) to show that the equivalent stress $\sigma_{EQ}$ may be written in the form of (*8.23*).

From equation (*8.24*),

$$\sigma_{EQ}^2 = 3s_{ij}s_{ij}/2 = 3(\sigma_{ij} - \delta_{ij}\sigma_{pp}/3)(\sigma_{ij} - \delta_{ij}\sigma_{qq}/3)/2 = (3\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj})/2$$

Expanding this gives

$$\begin{aligned}&[3(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2) + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) - (\sigma_{11} + \sigma_{22} + \sigma_{33})^2]/2 \\ &= [2(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - \sigma_{11}\sigma_{22} - \sigma_{22}\sigma_{33} - \sigma_{33}\sigma_{11}) + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2)]/2 \\ &= [(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2)]/2\end{aligned}$$

which confirms (*8.23*).

**8.16.** In plastic potential theory the plastic strain increment vector is normal to the loading (yield) surface at a regular point. If $[N_1, N_2, N_3]$ are direction numbers of the normal to the yield surface $f_1(\sigma_{ij})$, show that $d\epsilon_{\text{I}}^P/s_{\text{I}} = d\epsilon_{\text{II}}^P/s_{\text{II}} = d\epsilon_{\text{III}}^P/s_{\text{III}}$ under the Mises yield condition and flow law.

The condition of normality is expressed by $\mathbf{N} = \text{grad}\, f_1$ which requires $N_1/(\partial f_1/\partial \sigma_{\text{I}}) = N_2/(\partial f_1/\partial \sigma_{\text{II}}) = N_3/(\partial f_1/\partial \sigma_{\text{III}})$ for the Mises case where $f_1 = (\sigma_{\text{I}} - \sigma_{\text{II}})^2 + (\sigma_{\text{II}} - \sigma_{\text{III}})^2 + (\sigma_{\text{III}} - \sigma_{\text{I}})^2 - 2\sigma_Y^2 = 0$. Here $\partial f_1/\partial \sigma_{\text{I}} = 2(2\sigma_{\text{I}} - \sigma_{\text{II}} - \sigma_{\text{III}}) = 6s_{\text{I}}$, etc., and since the plastic strain increment vector is along the normal it follows that $d\epsilon_{\text{I}}^P/s_{\text{I}} = d\epsilon_{\text{II}}^P/s_{\text{II}} = d\epsilon_{\text{III}}^P/s_{\text{III}}$.

**8.17.** Determine the plastic strain increment ratios for (*a*) simple tension with $\sigma_{11} = \sigma_Y$, (*b*) biaxial stress with $\sigma_{11} = -\sigma_Y/\sqrt{3}$, $\sigma_{22} = \sigma_Y/\sqrt{3}$, $\sigma_{33} = \sigma_{12} = \sigma_{23} = \sigma_{13} = 0$, (*c*) pure shear with $\sigma_{12} = \sigma_Y/\sqrt{3}$.

(*a*) Here $\sigma_{11} = \sigma_{\text{I}} = \sigma_Y$, $\sigma_{\text{II}} = \sigma_{\text{III}} = 0$ and $s_{\text{I}} = 2\sigma_Y/3$, $s_{\text{II}} = s_{\text{III}} = -\sigma_Y/3$. Thus from Problem 8.16, $d\epsilon_{\text{I}}^P/2 = -d\epsilon_{\text{II}}^P/1 = -d\epsilon_{\text{III}}^P/1$.

(*b*) Here $\sigma_{\text{I}} = \sigma_Y/\sqrt{3}$, $\sigma_{\text{II}} = 0$, $\sigma_{\text{III}} = -\sigma_Y/\sqrt{3}$ and $s_{\text{I}} = \sigma_Y/\sqrt{3}$, $s_{\text{II}} = 0$, $s_{\text{III}} = -\sigma_Y/\sqrt{3}$. Thus $d\epsilon_{\text{I}}^P/1 = -d\epsilon_{\text{III}}^P/1$ and the third term is omitted since it is usually understood in the theory that if the denominator is zero the numerator will be zero too.

(*c*) Here $\sigma_{\text{I}} = \sigma_Y/\sqrt{3}$, $\sigma_{\text{II}} = 0$, $\sigma_{\text{III}} = -\sigma_Y/\sqrt{3}$ and again $d\epsilon_{\text{I}}^P/1 = -d\epsilon_{\text{III}}^P/1$.

**8.18.** Determine the plastic work increment $dW^P$ and the equivalent plastic strain increment $d\epsilon_{EQ}^P$ for the biaxial stress state $\sigma_{11} = -\sigma_Y/\sqrt{3}$, $\sigma_{22} = \sigma_Y/\sqrt{3}$, $\sigma_{33} = \sigma_{12} = \sigma_{23} = \sigma_{31} = 0$ if plastic deformation is controlled so that $d\epsilon_{\text{I}}^P = C$, a constant.

In principal-axis form, (*8.30*) becomes $dW^P = \sigma_{\text{I}}\, d\epsilon_{\text{I}}^P + \sigma_{\text{II}}\, d\epsilon_{\text{II}}^P + \sigma_{\text{III}}\, d\epsilon_{\text{III}}^P$; and for the stress state given, Problem 8.17 shows that $d\epsilon_{\text{I}}^P = -d\epsilon_{\text{III}}^P$, $d\epsilon_{\text{II}}^P = 0$; hence

$$dW^P = -\sigma_Y C/\sqrt{3} + (\sigma_Y/\sqrt{3})(-C) = -2C\sigma_Y/\sqrt{3}$$

From (*8.25*),

$$\begin{aligned}d\epsilon_{EQ}^P &= \{2[(d\epsilon_{\text{I}}^P - d\epsilon_{\text{II}}^P)^2 + (d\epsilon_{\text{II}}^P - d\epsilon_{\text{III}}^P)^2 + (d\epsilon_{\text{III}}^P - d\epsilon_{\text{I}}^P)^2]\}^{1/2}/3 \\ &= \{2[C^2 + C^2 + 4C^2]\}^{1/2}/3 = 2C/\sqrt{3}\end{aligned}$$

**8.19.** Verify (*8.32*) by showing that for a Prandtl-Reuss material the plastic work increment is $dW^P = \sigma_{EQ}\, d\epsilon^P_{EQ}$ as given in (*8.31*).

From (*8.30*), $dW^P = s_{ij}s_{ij}\, d\lambda$ for a Prandtl-Reuss material satisfying (*8.21*). But from (*8.27*), $d\lambda = 3\, d\epsilon^P_{EQ}/2\sigma_{EQ}$ for such a material and so $dW^P = (3s_{ij}s_{ij}/2)(d\epsilon^P_{EQ}/\sigma_{EQ})$ which because of the definition (*8.24*) gives $dW^P = \sigma_{EQ}\, d\epsilon^P_{EQ}$. Thus $d\epsilon^P_{EQ} = dW^P/\sigma_{EQ}$ here and (*8.32*) follows directly from (*8.21*).

**8.20.** For a material obeying the Mises yield condition, the equivalent stress $\sigma_{EQ}$ may be taken as the yield function in the hardening rules (*8.34*) and (*8.36*). Show that in this case $\sigma_{EQ}F' = H'$ where $F'$ and $H'$ are the derivatives of the hardening functions with respect to their respective arguments.

Here (*8.34*) becomes $\sigma_{EQ} = F(W^P)$ and so $d\sigma_{EQ} = F'\, dW^P$. Likewise (*8.36*) is given here by $\sigma_{EQ} = H(\epsilon^P_{EQ})$ and so $d\sigma_{EQ} = H'\, d\epsilon^P_{EQ}$. Thus $F'\, dW^P = H'\, d\epsilon^P_{EQ}$; and since from (*8.31*) (or Problem 8.19) $dW^P = \sigma_{EQ}\, d\epsilon^P_{EQ}$, it follows at once that $\sigma_{EQ}F' = H'$.

## TOTAL DEFORMATION THEORY (Sec. 8.9)

**8.21.** The Hencky total deformation theory may be represented through the equations $\epsilon_{ij} = \epsilon^E_{ij} + \epsilon^P_{ij}$ with $\epsilon^E_{ij} = e^E_{ij} + \delta_{ij}\epsilon^E_{kk}/3 = (s_{ij}/2)G + \delta_{ij}(1-2\nu)\sigma_{kk}/3E$ and $\epsilon^P_{ij} = \phi s_{ij}$. Show that these equations are equivalent to (*8.37*) and (*8.38*).

The equation $\epsilon^P_{ij} = \phi s_{ij}$ implies $\epsilon^P_{ii} = 0$ so that $\epsilon^P_{ij} = e^P_{ij} = \phi s_{ij}$, and from $\epsilon_{ij} = \epsilon^E_{ij} + \epsilon^P_{ij}$ it follows that here $\epsilon_{jj} = \epsilon^E_{jj}$. From the same equation, $e_{ij} + \delta_{ij}\epsilon_{kk}/3 = e^E_{ij} + \delta_{ij}\epsilon^E_{kk}/3 + e^P_{ij}$ which reduces to $e_{ij} = e^E_{ij} + e^P_{ij} = (\phi + \frac{1}{2}G)s_{ij}$, (*8.37*). Also from $\epsilon_{ii} = \epsilon^E_{ii}$, $\epsilon_{ii} = (1-2\nu)\sigma_{kk}/E$, (*8.38*).

**8.22.** Verify that the Hencky parameter $\phi$ may be expressed as given in (*8.39*).

Squaring and adding the components in the equation $\epsilon^P_{ij} = \phi s_{ij}$ of Problem 8.21 gives $\epsilon^P_{ij}\epsilon^P_{ij} = \phi^2 s_{ij}s_{ij}$ or $\phi = \sqrt{3\epsilon^P_{ij}\epsilon^P_{ij}/2}\,/\sigma_{EQ}$ which when multiplied on each side by 2/3 becomes

$$\phi = 3\sqrt{2\epsilon^P_{ij}\epsilon^P_{ij}/3}\,/2\sigma_{EQ} = 3\epsilon^P_{EQ}/2\sigma_{EQ}$$

## ELASTOPLASTIC PROBLEMS (Sec. 8.10)

**8.23.** An elastic-perfectly plastic rectangular beam is loaded steadily in pure bending. Using simple beam theory, determine the end moments $M$ for which the remaining elastic core extends from $-a$ to $a$ as shown in Fig. 8-15.

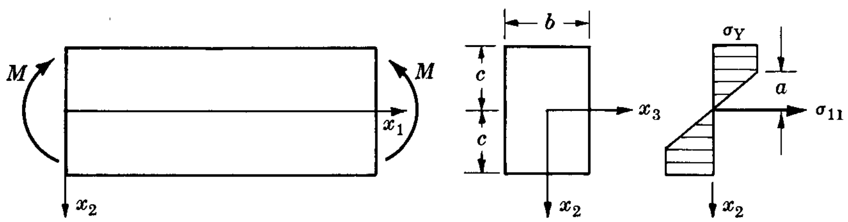



Fig. 8-15

Here the only nonzero stress is the bending stress $\sigma_{11}$. In the elastic portion of the beam $(-a < x_2 < a)$, $\sigma_{11} = E\epsilon_{11} = Ex_2/R$ where $R$ is the radius of curvature and $E$ is Young's modulus. In the plastic portion, $\sigma_{11} = \sigma_Y$. Thus

$$M = 2\int_0^a \frac{E}{R}(x_2)^2 b\, dx_2 + 2\int_a^c x_2\sigma_Y b\, dx_2 = b\sigma_Y(c^2 - a^2/3)$$

where $\sigma_Y = Ea/R$, the stress condition at the elastic-plastic interface, has been used. From the result obtained, $M = 2bc^2\sigma_Y/3$ at first yield (when $a = c$), and $M = bc^2\sigma_Y$ for the fully-plastic beam (when $a = 0$).

**8.24.** Determine the moment for a beam loaded as in Problem 8.23 if the material is a piecewise linear hardening material for which $\sigma_{11} = \sigma_Y + A(\epsilon_{11} - \sigma_Y/E)$ after yield.

The stress distribution for this beam is shown in Fig. 8-16. Again $\epsilon_{11} = x_2/R$ and so

$$M = 2\int_0^a \frac{E(x_2)^2 b}{R}\,dx_2 + 2\int_a^c \left[\sigma_Y + A\left(\frac{x_2}{R} - \frac{\sigma_Y}{E}\right)\right] x_2 b\,dx_2$$

$$= \frac{2Eba^3}{3R} + 2b\left\{\frac{\sigma_Y}{2}\left(1 - \frac{A}{E}\right)(c^2 - a^2) + \frac{A(c^3 - a^3)}{3R}\right\}$$

or using $\sigma_Y = Ea/R$ as in Problem 8.23,

$$M = c^2 b\sigma_Y(1 - A/E) + 2c^3 bA/3R + b\sigma_Y^3 R^2(A/E - 1)/3E^2$$

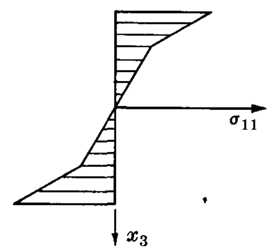


Fig. 8-16

**8.25.** An elastic-perfectly plastic circular shaft of radius $c$ is twisted by end torques $T$ as shown in Fig. 8-17. Determine the torque for which an inner elastic core of radius $a$ remains.

The shear stress $\sigma_{12}$ is given here by $\sigma_{12} = kr/a$ for $0 \leq r \leq a$, and by $\sigma_{12} = k$ for $a \leq r \leq c$ where $k$ is the yield stress of the material in shear. Thus

$$T = 2\pi\int_0^a (kr^3/a)\,dr + 2\pi\int_a^c kr^2\,dr = \frac{2\pi k}{3}(c^3 - a^3/4)$$

Therefore the torque at first yield is $T_1 = \pi kc^3/2$ when $a = c$; and for the fully plastic condition, $T_2 = 2\pi kc^3/3 = 4T_1/3$ when $a = 0$.

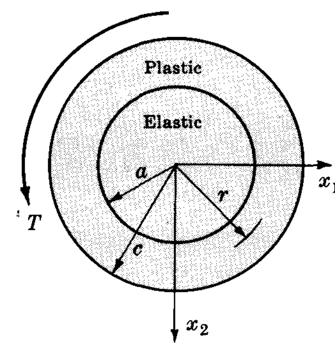


Fig. 8-17

**8.26.** A thick spherical shell of the dimensions shown in Fig. 8-18 is subjected to an increasing pressure $p_0$. Using the Mises yield condition, determine the pressure at which first yield occurs.

Because of symmetry of loading the principal stresses are the spherical components $\sigma_{(\theta\theta)} = \sigma_I = \sigma_{II}$, $\sigma_{(rr)} = \sigma_{III}$. Thus the Mises yield condition (*8.12*) becomes $\sigma_{(\theta\theta)} - \sigma_{(rr)} = \sigma_Y$. The elastic stress components may be shown to be

$$\sigma_{(rr)} = -p_0(b^3/r^3 - 1)/(b^3/a^3 - 1)$$

$$\sigma_{(\theta\theta)} = \sigma_{(\phi\phi)} = p_0(b^3/2r^3 + 1)/(b^3/a^3 - 1)$$

Therefore $\sigma_Y = 3b^3 p_0/2r^3(b^3/a^3 - 1)$ and $p_0 = 2\sigma_Y(1 - a^3/b^3)/3$ at first yield which occurs at the inner radius $a$.

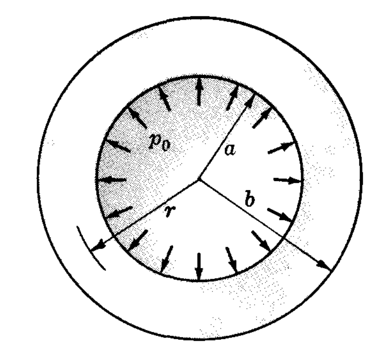


Fig. 8-18

## SLIP LINE THEORY (Sec. 8.11)

**8.27.** Verify directly the principal stress values (*8.45*) for the stress tensor (*8.41*) with $\sigma_{33} = (\sigma_{11} + \sigma_{22})/2$ as given in (*8.44*).

The principal stress values are found from the determinant equation (*2.37*) which here becomes

$$\begin{vmatrix} \sigma_{11} - \sigma & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} - \sigma & 0 \\ 0 & 0 & -p - \sigma \end{vmatrix} = 0$$

Expanding by the third column,

$$(-p-\sigma)[(\sigma_{11}-\sigma)(\sigma_{22}-\sigma)-\sigma_{12}^2] = (-p-\sigma)[\sigma^2-(\sigma_{11}+\sigma_{22})\sigma-\sigma_{12}^2] = 0$$

The roots of this equation are clearly $\sigma = -p$ and $\sigma = \frac{1}{2}(\sigma_{11}+\sigma_{22}) \pm \sqrt{\frac{1}{4}(\sigma_{11}+\sigma_{22})^2+\sigma_{12}^2} = -p \pm k$.

**8.28.** Using the condition that the yield stress in shear $k$ is constant, combine (*8.47*) with the equilibrium equations and integrate to prove (*8.48*).

From the equilibrium equations $\partial\sigma_{11}/\partial x_1 + \partial\sigma_{12}/\partial x_2 = 0$ and $\partial\sigma_{12}/\partial x_1 + \partial\sigma_{22}/\partial x_2 = 0$ which are valid here, (*8.47*) yields

$$-\partial p/\partial x_1 - k(2\cos 2\phi)(\partial\phi/\partial x_1) + k(-2\sin 2\phi)(\partial\phi/\partial x_2) = 0$$

and
$$-k(2\sin 2\phi)(\partial\phi/\partial x_1) - \partial p/\partial x_2 + k(2\cos 2\phi)(\partial\phi/\partial x_2) = 0$$

If $x_1$ is along an $\alpha$ line and $x_2$ along a $\beta$ line, $\phi = 0$ and these equations become $-\partial p/\partial x_1 - 2k(\partial\phi/\partial x_1) = 0$ along the $\alpha$ line, $-\partial p/\partial x_2 + 2k(\partial\phi/\partial x_2) = 0$ along the $\beta$ line. Integrating directly, $p + 2k\phi = C_1$ on the $\alpha$ line, $p - 2k\phi = C_2$ on the $\beta$ line.

**8.29.** In the frictionless extrusion through a square die causing a fifty per cent reduction, the centered fan region is composed of straight radial $\beta$ lines and circular $\alpha$ lines as shown in Fig. 8-19. Determine the velocity components along these slip lines in terms of the approach velocity $U$ and the polar coordinates $r$ and $\theta$.

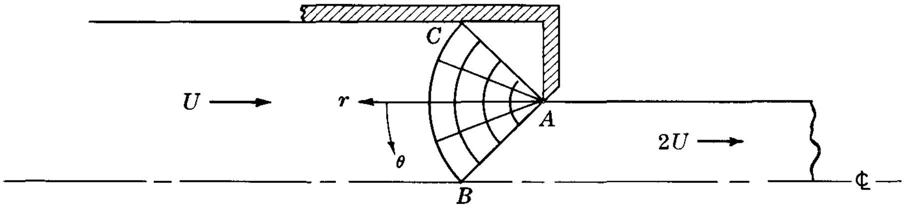



Fig. 8-19

Along the straight $\beta$ lines, $d\phi = 0$; and from (*8.53*), $dv_2 = 0$ or $v_2 =$ constant. From the normal velocity continuity along $BC$, the constant here must be $U\cos\theta$ and so $v_2 = U\cos\theta$. Along the circular $\alpha$ lines, $d\phi = d\theta$; and from (*8.52*),

$$v_1 = \int_{-\pi/4}^{\theta} U\cos\theta\, d\theta = U(\sin\theta + 1/\sqrt{2})$$

## MISCELLANEOUS PROBLEMS

**8.30.** Show that the von Mises yield condition may be expressed in terms of the octahedral shear stress $\sigma_{oct}$ (see Problem 2.22) by $\sigma_{oct} = \sqrt{2}\sigma_Y/3$.

In terms of principal stresses $3\sigma_{oct} = \sqrt{(\sigma_I-\sigma_{II})^2+(\sigma_{II}-\sigma_{III})^2+(\sigma_{III}-\sigma_I)^2}$ (Problem 2.22) and so $9\sigma_{oct}^2 = (\sigma_I-\sigma_{II})^2+(\sigma_{II}-\sigma_{III})^2+(\sigma_{III}-\sigma_I)^2 = 2\sigma_Y^2$ in agreement with (*8.12*).

**8.31.** Show that equation (*8.13*) for Mises yield condition may be written as

$$s_I^2 + s_{II}^2 + s_{III}^2 = 2k^2$$

From (*2.71*), $\sigma_I = s_I + \sigma_M$, etc., and so (*8.13*) at once becomes

$$(s_I - s_{II})^2 + (s_{II}-s_{III})^2 + (s_{III}-s_I)^2 = 6k^2$$

Expanding and rearranging, this may be written $s_I^2 + s_{II}^2 + s_{III}^2 - (s_I+s_{II}+s_{III})^2/3 = 2k^2$. But $s_I + s_{II} + s_{III} = I_{\Sigma_D} \equiv 0$ and the required equation follows.

**8.32.** At what value of the Lode parameter $\mu = (2\sigma_{II} - \sigma_I - \sigma_{III})/(\sigma_I - \sigma_{III})$ are the Tresca and Mises yield conditions identical?

From the definition of $\mu$, $\sigma_{II} = (\sigma_I + \sigma_{III})/2 + \mu(\sigma_I - \sigma_{III})/2$ which when substituted into the Mises yield condition (*8.12*) gives after some algebra (see Problem 8.42) $\sigma_I - \sigma_{III} = 2\sigma_Y/\sqrt{3+\mu^2}$. Tresca's yield condition, equation (*8.8*), is $\sigma_I - \sigma_{III} = \sigma_Y$. Thus when $\mu = 1$ the two are identical. When $\sigma_{II} = \sigma_I$, $\mu = 1$ which is sometimes called a cylindrical state of stress.

**8.33.** For the state of stress $\sigma_{ij} = \begin{pmatrix} \sigma & \tau & 0 \\ \tau & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}$ where $\sigma$ and $\tau$ are constants, determine the yield condition according to Tresca and von Mises criteria.

The principal stresses here are readily shown to be $\sigma_I = \sigma + \tau$, $\sigma_{II} = \sigma$, $\sigma_{III} = \sigma - \tau$. Thus from (*8.8*), the Tresca condition $\sigma_I - \sigma_{III} = \sigma_Y$ gives $2\tau = \sigma_Y$. From (*8.12*) the Mises condition gives $\tau = \sigma_Y/\sqrt{3}$. Note that in each case yielding depends on $\tau$, not on $\sigma$, i.e. yielding is independent of hydrostatic stress.

**8.34.** Show that the Prandtl-Reuss equations imply incompressible plastic deformation and write the equations in terms of actual stresses.

From (*8.21*), $d\epsilon_{ii}^P = s_{ii}\,d\lambda = 0$ since $s_{ii} = I_{\Sigma_D} \equiv 0$ and the incompressibility condition $d\epsilon_{ii}^P = 0$ is attained. In terms of stresses, $d\epsilon_{ij}^P = (\sigma_{ij} - \delta_{ij}\sigma_{kk}/3)\,d\lambda$. Thus $d\epsilon_{11}^P = (2/3)[\sigma_{11} - (\sigma_{22} + \sigma_{33})/2]\,d\lambda$, etc., for the normal components and $d\epsilon_{12}^P = \sigma_{12}\,d\lambda$, etc., for the shear components.

**8.35.** Using the von Mises yield condition, show that in the $\Pi$-plane the deviator stress components at yield are

$$s_I = [-2\sigma_Y \cos(\theta - \pi/6)]/3, \qquad s_{II} = [2\sigma_Y \cos(\theta + \pi/6)]/3, \qquad s_{III} = (2\sigma_Y \sin\theta)/3$$

where $\theta = \tan^{-1} Y/X$ in the notation of Problem 8.6.

The radius of the Mises yield circle is $\sqrt{2/3}\,\sigma_Y$ so that by definition $X = \sqrt{2/3}\,\sigma_Y \cos\theta$, $Y = \sqrt{2/3}\,\sigma_Y \sin\theta$ at yield. From the transformation table given in Problem 8.6 together with $\sigma_I = s_I + \sigma_M$, etc., the equations $s_I - s_{II} = -\sqrt{2}\,X = -(2/\sqrt{3})\sigma_Y \cos\theta$ and $s_I + s_{II} - 2s_{III} = -\sqrt{6}\,Y = -2\sigma_Y \sin\theta$ are obtained. Also, in the $\Pi$-plane, $s_I + s_{II} + s_{III} = 0$. Solving these three equations simultaneously yields the desired expressions, as the student should verify.

**8.36.** An elastic-perfectly plastic, incompressible material is loaded in plane strain between rigid plates so that $\sigma_{22} = 0$ and $\epsilon_{33} = 0$ (Fig. 8-20). Use Mises yield condition to determine the loading stress $\sigma_{11}$ at first yield, and the accompanying strain $\epsilon_{11}$.

The elastic stress-strain equation

$$E\epsilon_{33} = \sigma_{33} - \nu(\sigma_{11} + \sigma_{22})$$

reduces here to $\sigma_{33} = \nu\sigma_{11}$. Thus the principal stresses are $\sigma_I = 0$, $\sigma_{II} = -\nu\sigma_{11}$, $\sigma_{III} = -\sigma_{11}$; and by (*8.12*) we have

$$(\nu\sigma_{11})^2 + (\sigma_{11}(1-\nu))^2 + (-\sigma_{11})^2 = 2\sigma_Y^2$$

from which $\sigma_{11} = -\sigma_Y/\sqrt{1-\nu-\nu^2}$ (compressive) at yield. Likewise, from $E\epsilon_{11} = \sigma_{11} - \nu(\sigma_{22} + \sigma_{33})$ we see that here $\epsilon_{11} = -\sigma_Y(1-\nu^2)/E\sqrt{1-\nu-\nu^2}$ at yield.

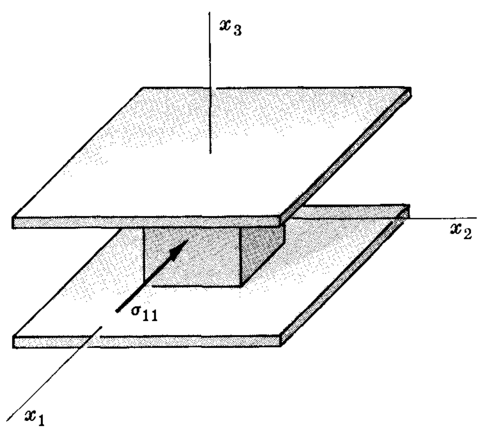



Fig. 8-20

**8.37.** An elastic-perfectly plastic rectangular beam is loaded in pure bending until fully plastic. Determine the residual stress in the beam upon removal of the bending moment $M$.

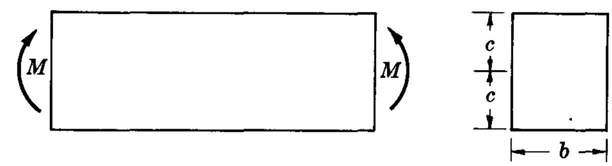



Fig. 8-21

For the fully plastic condition, the moment is (see Problem 8.23) $M = bc^2\sigma_Y$. This moment would cause an elastic stress having $\sigma = Mc/I = 3\sigma_Y/2$ at the extreme fibers, since $I = 2bc^3/3$. Thus removal of $M$ is equivalent to applying a corresponding negative elastic stress which results in the residual stress shown in Fig. 8-22.

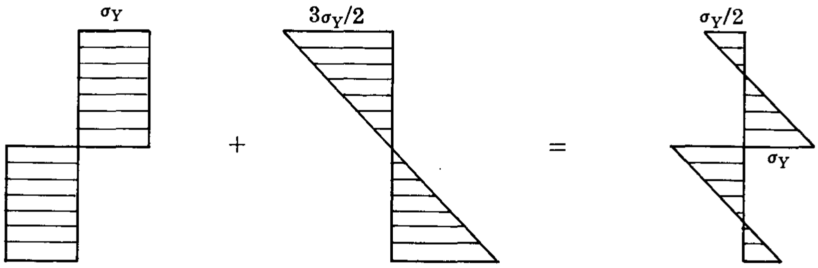



Fig. 8-22

**8.38.** A thick-walled cylindrical tube of the dimensions shown in Fig. 8-23 is subjected to an internal pressure $p_i$. Determine the value of $p_i$ at first yield if the ends of the tube are closed. Assume (*a*) von Mises and (*b*) Tresca's yield conditions.

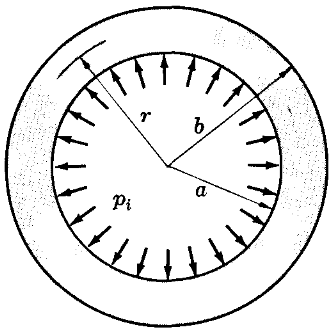



Fig. 8-23

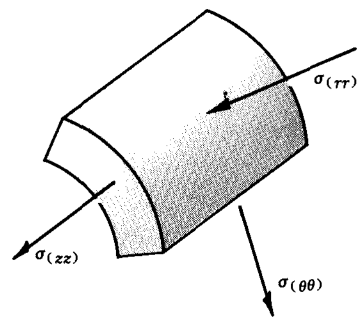



Fig. 8-24

The cylindrical stress components (Fig. 8-24) are principal stresses and for the elastic analysis may be shown to be $\sigma_{(rr)} = -p_i(b^2/r^2 - 1)/Q$, $\sigma_{(\theta\theta)} = p_i(b^2/r^2 + 1)/Q$, $\sigma_{(zz)} = p_i/Q$ where $Q = (b^2/a^2 - 1)$.

(*a*) Here Mises yield condition is

$$(\sigma_{(rr)} - \sigma_{(\theta\theta)})^2 + (\sigma_{(\theta\theta)} - \sigma_{(zz)})^2 + (\sigma_{(zz)} - \sigma_{(rr)})^2 = 2\sigma_Y^2 \quad \text{or} \quad p_i^2 b^4/r^4 = Q^2\sigma_Y^2/3$$

The maximum stress is at $r = a$, and at first yield $p_i = (\sigma_Y/\sqrt{3})(1 - a^2/b^2)$.

(*b*) For the Tresca yield condition, $\sigma_{(\theta\theta)} - \sigma_{(rr)} = \sigma_Y$ since $\sigma_{zz}$ is the intermediate principal stress. Thus $2p_i b^2/r^2 = Q\sigma_Y$ and now at $r = a$, $p_i = (\sigma_Y/2)(1 - a^2/b^2)$ at first yield.

# Supplementary Problems

8.39. A one-dimensional stress-strain law is given by $\sigma = K\epsilon^n$ where $K$ and $n$ are constants and $\epsilon$ is true strain. Show that the maximum load occurs at $\epsilon = n$.

8.40. Rework Problem 8.4 using the yield stress in shear $k$ in place of $\sigma_Y$ in the Mises and Tresca yield conditions. *Ans.* Mises: $(\sigma/\sqrt{3}\,k)^2 + (\tau/k)^2 = 1$; Tresca: $(\sigma/2k)^2 + (\tau/k)^2 = 1$

8.41. Making use of the material presented in Problem 8.6, verify the geometry of Fig. 8-5(*c*).

8.42. From the definition of Lode's parameter $\mu$ (see Problem 8.3) and the Mises yield condition, show that $\sigma_I - \sigma_{III} = 2\sigma_Y/\sqrt{3+\mu^2}$.

8.43. In the $\Pi$-plane where $\theta = \tan^{-1} Y/X$ with $X$ and $Y$ defined in Problem 8.6, show that $\mu = -\sqrt{3}\tan\theta$.

8.44. Show that the invariants of the deviator stress $II_{\Sigma_D} = s_{ij}s_{ij}/2$ and $III_{\Sigma_D} = s_{ij}s_{jk}s_{ki}/3$ may be written $II_{\Sigma_D} = (s_I^2 + s_{II}^2 + s_{III}^2)/2$ and $III_{\Sigma_D} = (s_I^3 + s_{II}^3 + s_{III}^3)/3$ respectively.

8.45. Show that von Mises yield condition may be written in the form

$$(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) = 6k^2$$

8.46. Following the procedure of Problem 8.17, determine the plastic strain increment ratios for (*a*) biaxial tension with $\sigma_{11} = \sigma_{22} = \sigma_Y$, (*b*) tension-torsion with $\sigma_{11} = \sigma_Y/2$, $\sigma_{12} = \sigma_Y/2$.
*Ans.* (*a*) $d\epsilon_{11}^P = d\epsilon_{22}^P = -d\epsilon_{33}^P/2$ (*b*) $d\epsilon_{11}^P/2 = -d\epsilon_{22}^P = -d\epsilon_{33}^P = d\epsilon_{12}^P/3$

8.47. Verify the following equivalent expressions for the effective plastic strain increment $d\epsilon_{EQ}^P$ and note that in each case $d\epsilon_{EQ}^P = d\epsilon_{11}^P$ for uniaxial tension $\sigma_{11}$.

(*a*) $d\epsilon_{EQ}^P = \sqrt{2/3}\,[(d\epsilon_{11}^P)^2 + (d\epsilon_{22}^P)^2 + (d\epsilon_{33}^P)^2 + 2(d\epsilon_{12}^P)^2 + 2(d\epsilon_{23}^P)^2 + 2(d\epsilon_{31}^P)^2]^{1/2}$

(*b*) $d\epsilon_{EQ}^P = (\sqrt{2}/3)[(d\epsilon_{11}^P - d\epsilon_{22}^P)^2 + (d\epsilon_{22}^P - d\epsilon_{33}^P)^2 + (d\epsilon_{33}^P - d\epsilon_{11}^P)^2 + 6(d\epsilon_{12}^P)^2 + 6(d\epsilon_{23}^P)^2 + 6(d\epsilon_{31}^P)^2]^{1/2}$

8.48. A thin-walled elastic-perfectly plastic tube is loaded in combined tension-torsion. An axial stress $\sigma = \sigma_Y/2$ is developed first and maintained constant while the shear stress $\tau$ is steadily increased from zero. At what value of $\tau$ will yielding first occur according to the Mises condition?
*Ans.* $\tau = \sigma_Y/2$

8.49. The beam of triangular cross section shown in Fig. 8-25 is subjected to pure bending. Determine the location of the neutral axis (a distance $b$ from top) of the beam when fully plastic. *Ans.* $b = h/\sqrt{2}$

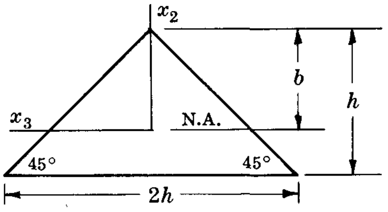


Fig. 8-25

8.50. Show that the stress tensor (*8.41*) becomes

$$\sigma_{ij} = \begin{pmatrix} -p & k & 0 \\ k & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$$

when referred to the axes rotated about $x_3$ by an angle $\theta$ in Fig. 8-8.

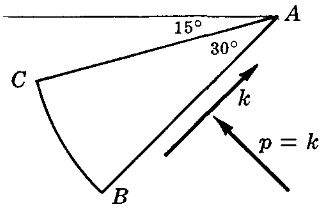


Fig. 8-26

8.51. A centered fan of $\alpha$ circle arcs and $\beta$ radii includes an angle of 30° as shown in Fig. 8-26. The pressure on $AB$ is $k$. Determine the pressure on $AC$.
*Ans.* $p = k(1 + \pi/3)$

# Chapter 9

## Linear Viscoelasticity

### 9.1 LINEAR VISCOELASTIC BEHAVIOR

Elastic solids and viscous fluids differ widely in their deformational characteristics. Elastically deformed bodies return to a natural or undeformed state upon removal of applied loads. Viscous fluids, however, possess no tendency at all for deformational recovery. Also, elastic stress is related directly to deformation whereas stress in a viscous fluid depends (except for the hydrostatic component) upon rate of deformation.

Material behavior which incorporates a blend of both elastic and viscous characteristics is referred to as *viscoelastic behavior*. The elastic (Hookean) solid and viscous (Newtonian) fluid represent opposite endpoints of a wide spectrum of viscoelastic behavior. Although viscoelastic materials are temperature sensitive, the discussion which follows is restricted to isothermal conditions and temperature enters the equations only as a parameter.

### 9.2 SIMPLE VISCOELASTIC MODELS

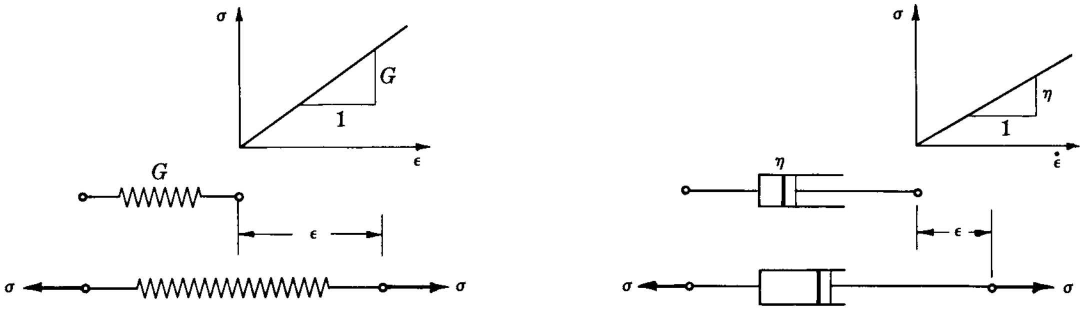
*Linear viscoelasticity* may be introduced conveniently from a one-dimensional viewpoint through a discussion of mechanical models which portray the deformational response of various viscoelastic materials. The mechanical elements of such models are the massless linear spring with spring constant $G$, and the viscous dashpot having a viscosity constant $\eta$. As shown in Fig. 9-1, the force across the spring $\sigma$ is related to its elongation $\epsilon$ by

$$\sigma = G\epsilon \tag{9.1}$$

and the analogous equation for the dashpot is given by

$$\sigma = \eta\dot{\epsilon} \tag{9.2}$$

where $\dot{\epsilon} = d\epsilon/dt$. The models are given more generality and dimensional effects removed by referring to $\sigma$ as *stress* and $\epsilon$ as *strain*, thereby putting these quantities on a per unit basis.



(*a*) Linear Spring (*b*) Viscous Dashpot

**Fig. 9-1**

The *Maxwell model* in viscoelasticity is the combination of a spring and dashpot in series as shown by Fig. 9-2(*a*). The *Kelvin* or *Voigt* model is the parallel arrangement shown in Fig. 9-2(*b*). The stress-strain relation (actually involving rates also) for the Maxwell model is

$$\frac{\dot{\sigma}}{G} + \frac{\sigma}{\eta} = \dot{\epsilon} \qquad (9.3)$$

and for the Kelvin model is

$$\sigma = G\epsilon + \eta\dot{\epsilon} \qquad (9.4)$$

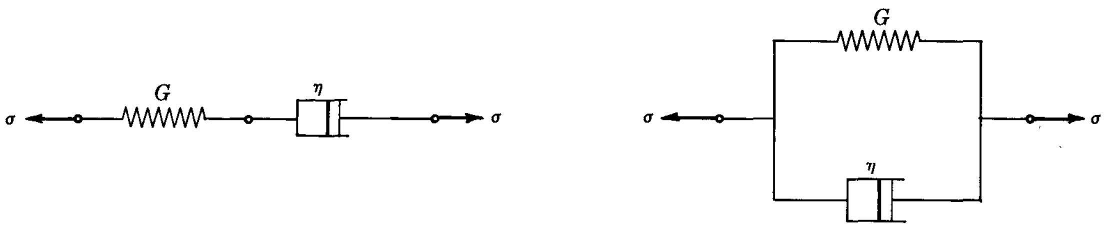
These equations are essentially one-dimensional viscoelastic constitutive equations. It is helpful to write them in operator form by use of the *linear differential time operator* $\partial_t \equiv \partial/\partial t$. Thus (*9.3*) becomes

$$\{\partial_t/G + 1/\eta\}\sigma = \{\partial_t\}\epsilon \qquad (9.5)$$

and (*9.4*) becomes

$$\sigma = \{G + \eta\partial_t\}\epsilon \qquad (9.6)$$

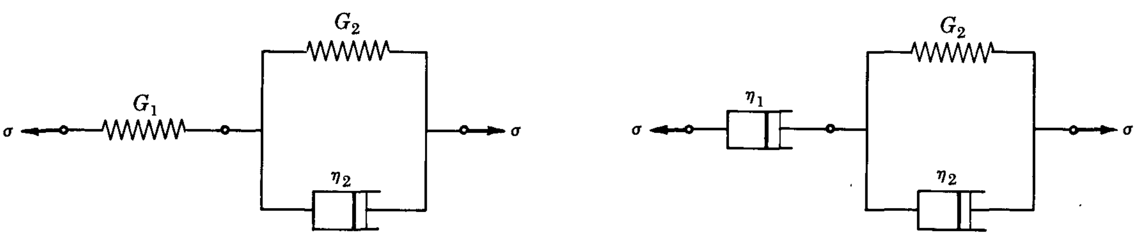
with the appropriate operators enclosed by parentheses.




(*a*) Maxwell (*b*) Kelvin

**Fig. 9-2**

The simple Maxwell and Kelvin models are not adequate to completely represent the behavior of real materials. More complicated models afford a greater flexibility in portraying the response of actual materials. A three-parameter model constructed from two springs and one dashpot, and known as the *standard linear solid* is shown in Fig. 9-3(*a*). A three-parameter viscous model consisting of two dashpots and one spring is shown in Fig. 9-3(*b*). It should be remarked that from the point of view of the form of their constitutive equations a Maxwell unit in parallel with a spring is analogous to the standard linear solid of Fig. 9-3(*a*), and a Maxwell unit in parallel with a dashpot is analogous to the viscous model of Fig. 9-3(*b*).




(*a*) Standard Linear Solid (*b*) Three-parameter Viscous Model

**Fig. 9-3**

A four-parameter model consisting of two springs and two dashpots may be regarded as a Maxwell unit in series with a Kelvin unit as illustrated in Fig. 9-4 below. Several equivalent forms of this model exist. The four-parameter model is capable of all three of the basic viscoelastic response patterns. Thus it incorporates "instantaneous elastic response" because of the free spring $G_1$, "viscous flow" because of the free dashpot $\eta_1$, and, finally, "delayed elastic response" from the Kelvin unit.

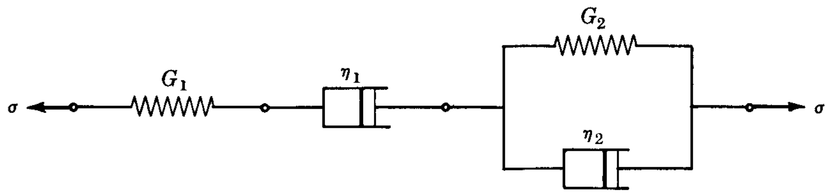



Fig. 9-4

The stress-strain equation for any of the three or four parameter models is of the general form

$$p_2\ddot{\sigma} + p_1\dot{\sigma} + p_0\sigma \;=\; q_2\ddot{\epsilon} + q_1\dot{\epsilon} + q_0\epsilon \tag{9.7}$$

where the $p_i$'s and $q_i$'s are coefficients made up of combinations of the $G$'s and $\eta$'s, and depend upon the specific arrangement of the elements in the model. In operator form, (*9.7*) is written

$$\{p_2\partial_t^2 + p_1\partial_t + p_0\}\sigma \;=\; \{q_2\partial_t^2 + q_1\partial_t + q_0\}\epsilon \tag{9.8}$$

## 9.3 GENERALIZED MODELS. LINEAR DIFFERENTIAL OPERATOR EQUATION

The *generalized Kelvin model* consists of a sequence of Kelvin units arranged in series as depicted by Fig. 9-5. The total strain of this model is equal to the sum of the individual Kelvin unit strains. Thus in operator form the constitutive equation is, by (*9.6*),

$$\epsilon \;=\; \frac{\sigma}{\{G_1 + \eta_1\partial_t\}} + \frac{\sigma}{\{G_2 + \eta_2\partial_t\}} + \cdots + \frac{\sigma}{\{G_N + \eta_N\partial_t\}} \tag{9.9}$$

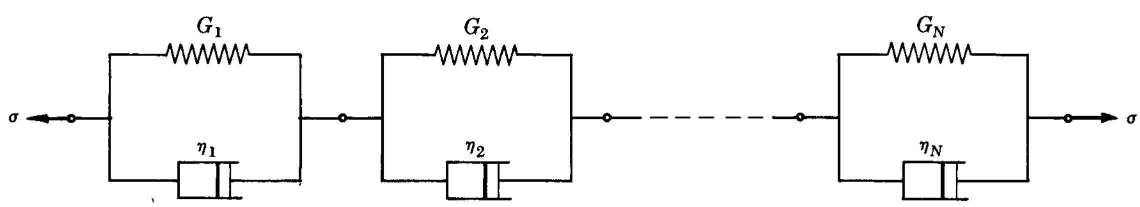



Fig. 9-5

Similarly, a sequence of Maxwell units in parallel as shown in Fig. 9-6 is called a *generalized Maxwell model.* Here the total stress is the resultant of the stresses across each unit; and so from (*9.5*),

$$\sigma \;=\; \frac{\dot{\epsilon}}{\{\partial_t/G_1 + 1/\eta_1\}} + \frac{\dot{\epsilon}}{\{\partial_t/G_2 + 1/\eta_2\}} + \cdots + \frac{\dot{\epsilon}}{\{\partial_t/G_N + 1/\eta_N\}} \tag{9.10}$$

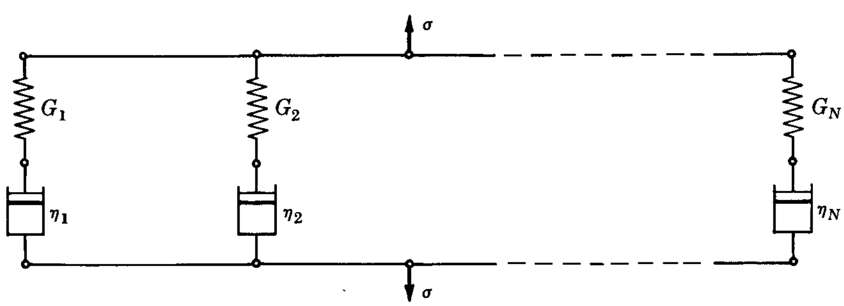



Fig. 9-6

For specific models, (*9.9*) and (*9.10*) result in equations of the form

$$p_0\sigma + p_1\dot{\sigma} + p_2\ddot{\sigma} + \cdots = q_0\epsilon + q_1\dot{\epsilon} + q_2\ddot{\epsilon} + \cdots \qquad (9.11)$$

which may be expressed compactly by

$$\sum_{i=0}^{m} p_i \frac{\partial^i \sigma}{\partial t^i} = \sum_{i=0}^{n} q_i \frac{\partial^i \epsilon}{\partial t^i} \qquad (9.12)$$

This *linear differential operator equation* may be written symbolically as

$$\{P\}\sigma = \{Q\}\epsilon \qquad (9.13)$$

where the operators $\{P\}$ and $\{Q\}$ are defined by

$$\{P\} = \sum_{i=0}^{m} p_i \frac{\partial^i}{\partial t^i}, \qquad \{Q\} = \sum_{i=0}^{n} q_i \frac{\partial^i}{\partial t^i} \qquad (9.14)$$

## 9.4 CREEP AND RELAXATION

The two basic experiments of viscoelasticity are the *creep* and *relaxation* tests. These tests may be performed as one-dimensional tension (compression) tests or as simple shear tests. The *creep experiment* consists of instantaneously subjecting a viscoelastic specimen to a stress $\sigma_0$ and maintaining the stress constant thereafter while measuring the strain (creep response) as a function of time. In the *relaxation experiment* an instantaneous strain $\epsilon_0$ is imposed and maintained on the specimen while measuring the stress (relaxation) as a function of time. Mathematically, the creep and relaxation loadings are expressed in terms of the *unit step function* $[U(t-t_1)]$, defined by

$$[U(t-t_1)] = \begin{cases} 1 & t < t_1 \\ 0 & t > t_1 \end{cases} \qquad (9.15)$$

and shown in Fig. 9-7.

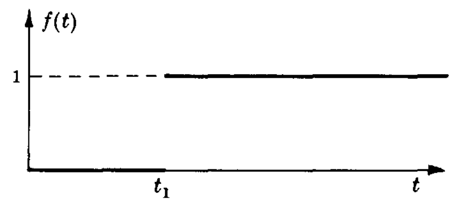


Fig. 9-7

For the creep loading,

$$\sigma = \sigma_0[U(t)] \qquad (9.16)$$

where $[U(t)]$ represents the unit step function applied at time $t_1 = 0$. The creep response of a Kelvin material is determined by solving the differential equation

$$\dot{\epsilon} + \frac{\epsilon}{\tau} = \frac{\sigma_0[U(t)]}{\eta} \qquad (9.17)$$

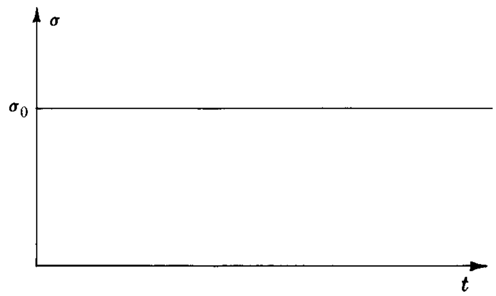
which results from the introduction of (*9.16*) into (*9.4*). Here $\tau = \eta/G$ is called the *retardation time.* For any continuous function of time $f(t)$, it may be shown that with $t'$ as the variable of integration,

$$\int_{-\infty}^{t} f(t')[U(t'-t_1)]\,dt' = [U(t-t_1)]\int_{t_1}^{t} f(t')\,dt' \qquad (9.18)$$

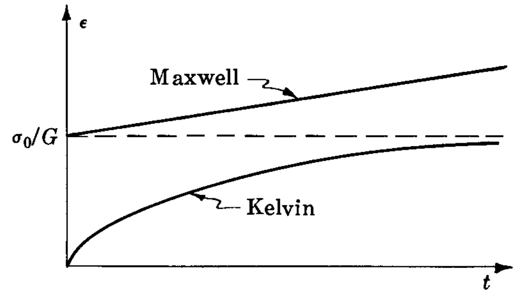
by means of which (*9.17*) may be integrated to yield the Kelvin creep response

$$\epsilon(t) = \frac{\sigma_0}{G}(1-e^{-t/\tau})[U(t)] \qquad (9.19)$$

The creep loading, together with the creep response for the Kelvin and Maxwell models (materials) is shown in Fig. 9-8 below.




(a) Creep Loading




(b) Creep Response

**Fig. 9-8**

The stress relaxation which occurs in a Maxwell material upon application of the strain

$$\epsilon = \epsilon_0[U(t)] \tag{9.20}$$

is given by the solution of the differential equation

$$\dot{\sigma} + \sigma/\tau = G\epsilon_0[\delta(t)] \tag{9.21}$$

obtained by inserting the time derivative of (*9.20*) into (*9.3*). Here $[\delta(t)] = d[U(t)]/dt$ is a singularity function called the *unit impulse function*, or *Dirac delta function*. By definition,

$$[\delta(t-t_1)] = 0, \qquad t \neq t_1 \tag{9.22a}$$

$$\int_{-\infty}^{\infty} [\delta(t-t_1)]\,dt = 1 \tag{9.22b}$$

This function is zero everywhere except at $t = t_1$ where it is said to have an indeterminate spike. For a continuous function $f(t)$, it may be shown that when $t > t_1$,

$$\int_{-\infty}^{t} f(t')\,[\delta(t'-t_1)]\,dt' = f(t_1)\,[U(t-t_1)] \tag{9.23}$$

with the help of which (*9.21*) may be integrated to give the Maxwell stress relaxation

$$\sigma(t) = G\epsilon_0 e^{-t/\tau}[U(t)] \tag{9.24}$$

The stress relaxation for a Kelvin material is given directly by inserting $\dot{\epsilon} = \epsilon_0[\delta(t)]$ into (*9.4*) to yield

$$\sigma(t) = G\epsilon_0[U(t)] + \eta\epsilon_0[\delta(t)] \tag{9.25}$$

The delta function in (*9.25*) indicates that it would require an infinite stress to produce an instantaneous finite strain in a Kelvin body.

## 9.5 CREEP FUNCTION. RELAXATION FUNCTION. HEREDITARY INTEGRALS

The creep response of any material (model) to the creep loading $\sigma = \sigma_0[U(t)]$ may be written in the form

$$\epsilon(t) = \Psi(t)\sigma_0 \tag{9.26}$$

where $\Psi(t)$ is known as the *creep function*. For example, the creep function for the generalized Kelvin model of Fig. 9-5 is determined from (*9.19*) to be

$$\Psi(t) = \sum_{i=1}^{N} J_i(1 - e^{-t/\tau_i})[U(t)] \tag{9.27}$$

where $J_i = 1/G_i$ is called the *compliance.* If the number of Kelvin units increases indefinitely so that $N \to \infty$ in such a way that the finite set of constants $(\tau_i, J_i)$ may be replaced by the continuous compliance function $J(\tau)$, the Kelvin creep function becomes

$$\Psi(t) = \int_0^\infty J(\tau)(1 - e^{-t/\tau})\,d\tau \tag{9.28}$$

The function $J(\tau)$ is referred to as the *"distribution of retardation times"*, or *retardation spectrum.*

In analogy with the creep response, the stress relaxation for any model subjected to the strain $\epsilon = \epsilon_0[U(t)]$ may be written in the form

$$\sigma(t) = \phi(t)\epsilon_0 \tag{9.29}$$

where $\phi(t)$ is called the *relaxation function.* For the generalized Maxwell model of Fig. 9-6, the relaxation function is determined from (*9.24*) as

$$\phi(t) = \sum_{i=1}^{N} G_i e^{-t/\tau_i}[U(t)] \tag{9.30}$$

Here, as $N \to \infty$ the function $G(\tau)$ replaces the set of constants $(G_i, \tau_i)$ and the relaxation function is defined by

$$\phi(t) = \int_0^\infty G(\tau)e^{-t/\tau}\,d\tau \tag{9.31}$$

The function $G(\tau)$ is known as the *"distribution of relaxation times"*, or *relaxation spectrum.*

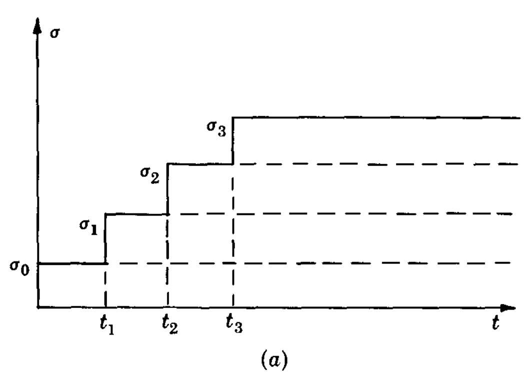
In linear viscoelasticity, the superposition principle is valid. Thus the total "effect" of a sum of "causes" is equal to the sum of the "effects" of each of the "causes". Accordingly, if the stepped stress history of Fig. 9-9(*a*) is applied to a material for which the creep function is $\Psi(t)$, the creep response will be

$$\epsilon(t) = \sigma_0\Psi(t) + \sigma_1\Psi(t - t_1) + \sigma_2\Psi(t - t_2) + \sigma_3\Psi(t - t_3) = \sum_{i=0}^{3} \sigma_i\Psi(t - t_i) \tag{9.32}$$

Therefore the arbitrary stress history $\sigma = \sigma(t)$ of Fig. 9-9(*b*) may be analyzed as an infinity of step loadings, each of magnitude $d\sigma$ and the creep response given by the superposition integral

$$\epsilon(t) = \int_{-\infty}^{t} \frac{d\sigma(t')}{dt'}\,\Psi(t - t')\,dt' \tag{9.33}$$

Such integrals are often referred to as *hereditary integrals* since the strain at any time is seen to depend upon the entire stress history.



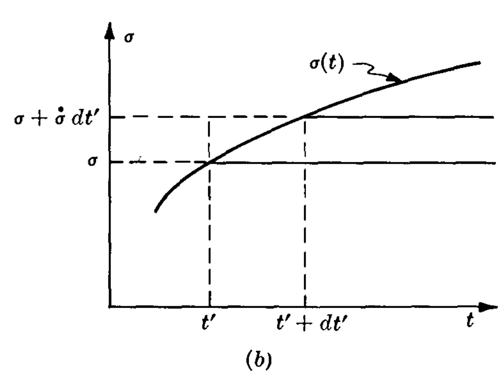




**Fig. 9-9**

For a material initially "dead", i.e. completely free of stress and strain at time zero, the lower limit in (*9.33*) may be replaced by zero and the creep response expressed as

$$\epsilon(t) = \int_0^t \frac{d\sigma(t')}{dt'} \Psi(t-t')\,dt' \tag{9.34}$$

Furthermore, if the stress loading involves a step discontinuity of magnitude $\sigma_0$ at $t=0$, (*9.34*) is usually written in the form

$$\epsilon(t) = \sigma_0\Psi(t) + \int_0^t \frac{d\sigma(t')}{dt'} \Psi(t-t')\,dt' \tag{9.35}$$

Following similar arguments as above, the stress as a function of time may be represented through a superposition integral involving the strain history $\epsilon(t)$ and the relaxation function $\phi(t)$. In analogy with (*9.33*) the stress is given by

$$\sigma(t) = \int_{-\infty}^t \frac{d\epsilon(t')}{dt'} \phi(t-t')\,dt' \tag{9.36}$$

and with regard to a material that is "dead" at $t=0$, the integrals comparable to (*9.34*) and (*9.35*) are respectively

$$\sigma(t) = \int_0^t \frac{d\epsilon(t')}{dt'} \phi(t-t')\,dt' \tag{9.37}$$

and

$$\sigma(t) = \epsilon_0\phi(t) + \int_0^t \frac{d\epsilon(t')}{dt'} \phi(t-t')\,dt' \tag{9.38}$$

Since either the creep integral (*9.34*) or the relaxation integral (*9.37*) may be used to specify the viscoelastic characteristics of a given material, it follows that some relationship must exist between the creep function $\Psi(t)$ and the relaxation function $\phi(t)$. Such a relationship is not easily determined in general, but using the Laplace transform definition

$$\bar{f}(s) = \int_0^\infty f(t)e^{-st}\,dt \tag{9.39}$$

it is possible to show that the transforms $\bar{\Psi}(s)$ and $\bar{\phi}(t)$ are related by the equation

$$\bar{\Psi}(s)\bar{\phi}(s) = 1/s^2 \tag{9.40}$$

where $s$ is the transform parameter.

## 9.6 COMPLEX MODULI AND COMPLIANCES

If a linearly viscoelastic test specimen is subjected to a one-dimensional (tensile or shear) stress loading $\sigma = \sigma_0 \sin \omega t$, the resulting steady state strain will be $\epsilon = \epsilon_0 \sin(\omega t - \delta)$, a sinusoidal response of the same frequency $\omega$ but out of phase with the stress by the lag angle $\delta$. The stress and strain for this situation may be presented graphically by the vertical projections of the constant magnitude vectors rotating at a constant angular velocity $\omega$ as shown in Fig. 9-10 below.

The ratios of the stress and strain amplitudes define the *absolute dynamic modulus* $\sigma_0/\epsilon_0$, and the *absolute dynamic compliance* $\epsilon_0/\sigma_0$. In addition, the in-phase and out-of-phase components of the stress and strain rotating vectors of Fig. 9-10(*a*) are used to define

(*a*) the storage modulus $\qquad G_1 = \dfrac{\sigma_0 \cos\delta}{\epsilon_0}$

(*b*) the loss modulus $$G_2 = \frac{\sigma_0 \sin \delta}{\epsilon_0}$$

(*c*) the storage compliance $$J_1 = \frac{\epsilon_0 \cos \delta}{\sigma_0}$$

(*d*) the loss compliance $$J_2 = \frac{\epsilon_0 \sin \delta}{\sigma_0}$$

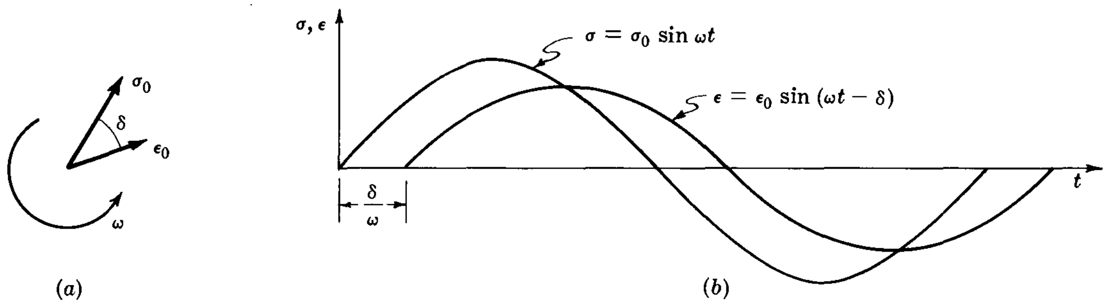



Fig. 9-10

A generalization of the above description of viscoelastic behavior is achieved by expressing the stress in complex form as

$$\sigma^* = \sigma_0 e^{i\omega t} \tag{9.41}$$

and the resulting strain also in complex form as

$$\epsilon^* = \epsilon_0 e^{i(\omega t - \delta)} \tag{9.42}$$

From (*9.41*) and (*9.42*) the *complex modulus* $G^*(i\omega)$ is defined as the complex quantity

$$\sigma^*/\epsilon^* = G^*(i\omega) = (\sigma_0/\epsilon_0)e^{i\delta} = G_1 + iG_2 \tag{9.43}$$

whose real part is the storage modulus and whose imaginary part is the loss modulus. Similarly, the *complex compliance* is defined as

$$\begin{aligned} \epsilon^*/\sigma^* &= J^*(i\omega) = (\epsilon_0/\sigma_0)e^{-i\delta} \\ &= J_1 - iJ_2 \end{aligned} \tag{9.44}$$

where the real part is the storage compliance and the imaginary part the negative of the loss compliance. In Fig. 9-11 the vector diagrams of $G^*$ and $J^*$ are shown. Note that $G^* = 1/J^*$.

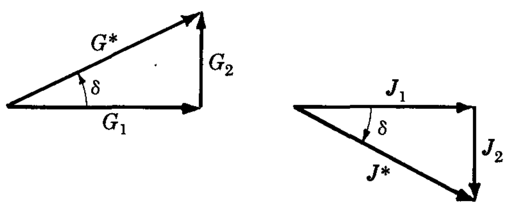



Fig. 9-11

## 9.7 THREE DIMENSIONAL THEORY

In developing the three dimensional theory of linear viscoelasticity, it is customary to consider separately viscoelastic behavior under conditions of so-called pure shear and pure dilatation. Thus distortional and volumetric effects are prescribed independently, and subsequently combined to provide a general theory. Mathematically, this is handled by resolving the stress and strain tensors into their deviatoric and spherical parts, for each of which viscoelastic constitutive relations are then written. The stress tensor decomposition is given by (*2.70*) as

$$\sigma_{ij} = s_{ij} + \delta_{ij}\sigma_{kk}/3 \tag{9.45}$$

and the small strain tensor by (*3.98*) as

$$\epsilon_{ij} = e_{ij} + \delta_{ij}\epsilon_{kk}/3 \tag{9.46}$$

Using the notation of these equations, the three dimensional generalization of the viscoelastic constitutive equation (*9.13*) in differential operator form is written by the combination

$$\{P\}s_{ij} = 2\{Q\}e_{ij} \tag{9.47a}$$

and

$$\{M\}\sigma_{ii} = 3\{N\}\epsilon_{ii} \tag{9.47b}$$

where $\{P\}$, $\{Q\}$, $\{M\}$ and $\{N\}$ are operators of the form (*9.14*) and the numerical factors are inserted for convenience. Since practically all materials respond elastically to moderate hydrostatic loading, the dilatational operators $\{M\}$ and $\{N\}$ are usually taken as constants and (*9.47*) modified to read

$$\{P\}s_{ij} = 2\{Q\}e_{ij} \tag{9.48a}$$

$$\sigma_{ii} = 3K\epsilon_{ii} \tag{9.48b}$$

where $K$ is the elastic bulk modulus.

Following the same general rule of separation for distortional and volumetric behavior, the three-dimensional viscoelastic constitutive relations in *creep integral* form are given by

$$e_{ij} = \int_0^t \Psi_s(t-t')\frac{\partial s_{ij}}{\partial t'}\,dt' \tag{9.49a}$$

$$\epsilon_{ii} = \int_0^t \Psi_v(t-t')\frac{\partial \sigma_{ii}}{\partial t'}\,dt' \tag{9.49b}$$

and in the *relaxation integral* form by

$$s_{ij} = \int_0^t \phi_s(t-t')\frac{\partial e_{ij}}{\partial t'}\,dt' \tag{9.50a}$$

$$\sigma_{ii} = \int_0^t \phi_v(t-t')\frac{\partial \epsilon_{ii}}{\partial t'}\,dt' \tag{9.50b}$$

The extension to three-dimensions of the complex modulus formulation of viscoelastic behavior requires the introduction of the complex bulk modulus $K^*$. Again, writing shear and dilatation equations separately, the appropriate equations are of the form

$$s_{ij}^* = 2G^*(i\omega)e_{ij}^* = 2(G_1 + iG_2)e_{ij}^* \tag{9.51a}$$

$$\sigma_{ii}^* = 3K^*(i\omega)\epsilon_{ii}^* = 3(K_1 + iK_2)\epsilon_{ii}^* \tag{9.51b}$$

## 9.8 VISCOELASTIC STRESS ANALYSIS. CORRESPONDENCE PRINCIPLE

The stress analysis problem for an isotropic viscoelastic continuum body which occupies a volume $V$ and has the bounding surface $S$ as shown in Fig. 9-12, is formulated as follows: Let body forces $b_i$ be given throughout $V$ and let the surface tractions $t_i^{(\hat{n})}(x_k, t)$ be prescribed over the portion $S_1$ of $S$, and the surface displacements $g_i(x_k, t)$ be prescribed over the portion $S_2$ of $S$. Then the governing field equations take the form of:

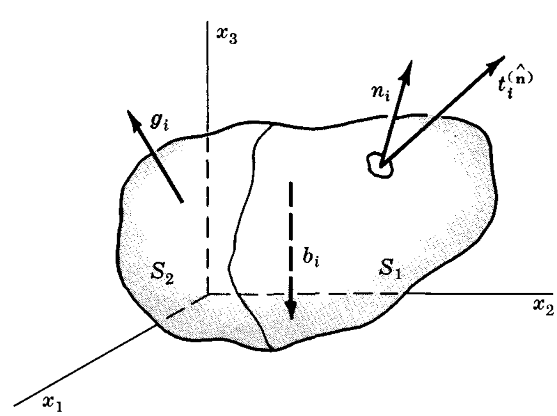



Fig. 9-12

1. Equations of motion (or of equilibrium)

$$\sigma_{ij,j} + b_i = \rho \ddot{u}_i \tag{9.52}$$

2. Strain-displacement equations

$$2\epsilon_{ij} = (u_{i,j} + u_{j,i}) \tag{9.53}$$

or strain-rate-velocity equations

$$2\dot{\epsilon}_{ij} = (v_{i,j} + v_{j,i}) \tag{9.54}$$

3. Boundary conditions

$$\sigma_{ij}(x_k, t)\, n_i(x_k) = t_i^{(\hat{n})}(x_k, t) \quad \text{on } S_1 \tag{9.55}$$

$$u_i(x_k, t) = g_i(x_k, t) \quad \text{on } S_2 \tag{9.56}$$

4. Initial conditions

$$u_i(x_k, 0) = u_0 \tag{9.57}$$

$$v_i(x_k, 0) = v_0 \tag{9.58}$$

5. Constitutive equations

(*a*) Linear differential operator form (*9.48*)

or

(*b*) Hereditary integral form (*9.49*) or (*9.50*)

or

(*c*) Complex modulus form (*9.51*)

If the body geometry and loading conditions are sufficiently simple, and if the material behavior may be represented by one of the simpler models, the field equations above may be integrated directly (see Problem 9.22). For more general conditions, however, it is conventional to seek a solution through the use of the *correspondence principle*. This principle emerges from the analogous form between the governing field equations of elasticity and the Laplace transforms with respect to time of the basic viscoelastic field equations given above. A comparison of the pertinent equations for quasi-static isothermal problems is afforded by the following table in which barred quantities indicate Laplace transforms in accordance with the definition

$$\bar{f}(x_k, s) = \int_0^\infty f(x_k, t) e^{-st}\, dt \tag{9.59}$$

| Elastic | Transformed Viscoelastic |
|---|---|
| 1. $\sigma_{ij,j} + b_i = 0$ | 1. $\bar{\sigma}_{ij,j} + \bar{b}_i = 0$ |
| 2. $2\epsilon_{ij} = (u_{i,j} + u_{j,i})$ | 2. $2\bar{\epsilon}_{ij} = (\bar{u}_{i,j} + \bar{u}_{j,i})$ |
| 3. $\sigma_{ij} n_j = t_i^{(\hat{n})}$ on $S_1$ | 3. $\bar{\sigma}_{ij} \bar{n}_j = \bar{t}_i^{(\hat{n})}$ on $S_1$ |
| $u_i = g_i$ on $S_2$ | $\bar{u}_i = \bar{g}_i$ on $S_2$ |
| 4. $s_{ij} = 2G e_{ij}$ | 4. $\bar{P}(s) \bar{s}_{ij} = 2\bar{Q}(s) \bar{e}_{ij}$ |
| $\sigma_{ii} = 3K\epsilon_{ii}$ | $\bar{\sigma}_{ii} = 3K\bar{\epsilon}_{ii}$ |

From this table it is observed that when $G$ in the elastic equations is replaced by $\bar{Q}/\bar{P}$, the two sets of equations have the same form. Accordingly, if in the solution of the "corresponding elastic problem" $G$ is replaced by $\bar{Q}/\bar{P}$ for the viscoelastic material involved, the result is the Laplace transform of the viscoelastic solution. Inversion of the transformed solution yields the viscoelastic solution.

The correspondence principle may also be stated for problems other than quasi-static problems. Furthermore, the form of the constitutive equations need not be the linear differential operator form but may appear as in (*9.49*), (*9.50*) or (*9.51*). The particular problem under study will dictate the appropriate form in which the principle should be used.

# Solved Problems

## VISCOELASTIC MODELS (Sec. 9.1-9.3)

**9.1.** Verify the stress-strain relations for the Maxwell and Kelvin models given by (*9.3*) and (*9.4*) respectively.

In the Maxwell model of Fig. 9-2(*a*) the total strain is the sum of the strain in the spring plus the strain of the dashpot. Thus $\epsilon = \epsilon_S + \epsilon_D$ and also $\dot{\epsilon} = \dot{\epsilon}_S + \dot{\epsilon}_D$. Since the stress across each element is $\sigma$, (*9.1*) and (*9.2*) may be used to give $\dot{\epsilon} = \dot{\sigma}/G + \sigma/\eta$.

In the Kelvin model of Fig. 9-2(*b*), $\sigma = \sigma_S + \sigma_D$ and directly from (*9.1*) and (*9.2*), $\sigma = \eta\dot{\epsilon} + G\epsilon$.

**9.2.** Use the operator form of the Kelvin model stress-strain relation to obtain the stress-strain law for the standard linear solid of Fig. 9-3(*a*).

Here the total strain is the sum of the strain in the spring plus the strain in the Kelvin unit. Thus $\epsilon = \epsilon_S + \epsilon_K$ or in operator form $\epsilon = \sigma/G_1 + \sigma/\{G_2 + \eta_2\partial_t\}$. From this

$$G_1\{G_2 + \eta_2\partial_t\}\epsilon = \{G_2 + \eta_2\partial_t\}\sigma + G_1\sigma$$

and so $G_1G_2\epsilon + G_1\eta_2\dot{\epsilon} = (G_1 + G_2)\sigma + \eta_2\dot{\sigma}$.

**9.3.** Determine the stress-strain equation for the four parameter model of Fig. 9-4. Let $\eta_1 \to \infty$ and compare with the result of Problem 9.2.

Here the total strain $\epsilon = \epsilon_K + \epsilon_M$ which in operator form is

$$\epsilon = \sigma/\{G_2 + \eta_2\partial_t\} + \{\partial_t + 1/\tau_1\}\sigma/G_1\{\partial_t\}$$

Expanding the operators and collecting terms gives

$$\ddot{\sigma} + (G_1/\eta_1 + (G_1 + G_2)/\eta_2)\dot{\sigma} + G_1G_2\sigma/\eta_1\eta_2 = G_1\ddot{\epsilon} + G_1G_2\dot{\epsilon}/\eta_2$$

As $\eta_1 \to \infty$ this becomes $\ddot{\sigma} + (G_1 + G_2)\dot{\sigma}/\eta_2 = G_1\ddot{\epsilon} + G_1G_2\dot{\epsilon}/\eta_2$ which is equivalent to the result of Problem 9.2.

**9.4.** Treating the model in Fig. 9-13 as a special case of the generalized Maxwell model, determine its stress-strain equation.

Writing (*9.10*) for $N = 2$ in the form

$$\sigma = G_1\dot{\epsilon}/\{\partial_t + 1/\tau_1\} + G_2\dot{\epsilon}/\{\partial_t + 1/\tau_2\}$$

and operating as indicated gives

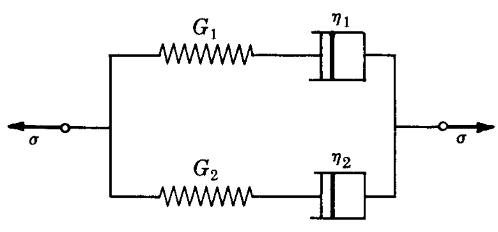



**Fig. 9-13**

$$\{\partial_t + 1/\tau_2\}(\dot{\sigma} + \sigma/\tau_1) = G_1\{\partial_t + 1/\tau_2\}\dot{\epsilon} + G_2\{\partial_t + 1/\tau_1\}\dot{\epsilon}$$

which when expanded and rearranged becomes

$$\ddot{\sigma} + (\tau_1 + \tau_2)\dot{\sigma}/\tau_1\tau_2 + \sigma/\tau_1\tau_2 = (G_1 + G_2)\ddot{\epsilon} + (G_1/\tau_2 + G_2/\tau_1)\dot{\epsilon}$$

**9.5.** The model shown in Fig. 9-14 may be considered as a degenerate form of the generalized Maxwell model with $G_1 = \eta_2 = \infty$ for the case $N = 3$. Using these values in (*9.10*) develop the stress-strain equation for this model.

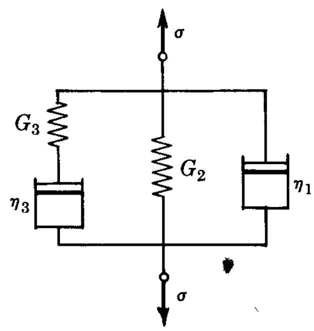


Fig. 9-14

Here (*9.10*) becomes $\sigma = \eta_1\dot{\epsilon} + G_2\dot{\epsilon}/\{\partial_t\} + \dot{\epsilon}/\{\partial_t/G_3 + 1/\eta_3\}$ or

$$\{\partial_t/G_3 + 1/\eta_3\}\dot{\sigma} = \{\partial_t/G_3 + 1/\eta_3\}(\eta_1\ddot{\epsilon} + G_2\dot{\epsilon}) + \dot{\epsilon}$$

Application of the operators gives

$$\ddot{\sigma}/G_3 + \dot{\sigma}/\eta_3 = \eta_1\dddot{\epsilon}/G_3 + (1 + G_2/G_3 + \eta_1/\eta_3)\ddot{\epsilon} + G_2/\eta_3\dot{\epsilon}$$

which may also be written

$$\eta_3\dot{\sigma} + G_3\sigma = \eta_1\eta_3\ddot{\epsilon} + (G_2\eta_3 + G_3\eta_1 + G_3\eta_3)\dot{\epsilon} + G_2G_3\epsilon$$

## CREEP AND RELAXATION (Sec. 9.4)

**9.6.** Determine the Kelvin and Maxwell creep response equations by direct integration of (*9.17*) and (*9.21*) respectively.

Using the integrating factor $e^{t/\tau}$, (*9.17*) becomes $\epsilon e^{t/\tau} = \frac{\sigma_0}{\eta}\int_0^t e^{t'/\tau}[U(t')]\,dt'$ which by formula (*9.18*) yields

$$\epsilon e^{t/\tau} = (\sigma_0[U(t)]/\eta)[\tau e^{t'/\tau}]_0^t = (\sigma_0/G)(e^{t/\tau} - 1)[U(t)] \quad \text{or} \quad \epsilon = (\sigma_0/G)(1 - e^{-t/\tau})[U(t)]$$

Use of $e^{t/\tau}$ as the integrating factor in (*9.21*) gives $\sigma e^{t/\tau} = G\epsilon_0\int_0^t e^{t'/\tau}[\delta(t')]\,dt'$; and by formula (*9.23*),

$$\sigma e^{t/\tau} = G\epsilon_0[U(t)] \quad \text{or} \quad \sigma = G\epsilon_0 e^{-t/\tau}[U(t)]$$

**9.7.** Determine the creep response of the standard linear solid of Fig. 9-3(*a*).

Since $\epsilon = \epsilon_S + \epsilon_K$ for this model the creep response from (*9.1*) and (*9.19*) is simply

$$\epsilon(t) = [1/G_1 + (1/G_2)(1 - e^{-t/\tau_2})]\sigma_0[U(t)]$$

The same result may be obtained by setting $\eta_2 = \infty$ in the generalized Kelvin $(N = 2)$ response $\epsilon = \sum_{i=1}^{2} J_i(1 - e^{-t/\tau_i})\sigma_0[U(t)]$ or by integrating directly the standard solid stress-strain law. The student should carry out the details.

**9.8.** The creep-recovery experiment consists of a creep loading which is maintained for a period of time and then instantaneously removed. Determine the creep-recovery response of the standard solid (Fig. 9-3(*a*)) for the loading shown in Fig. 9-15.

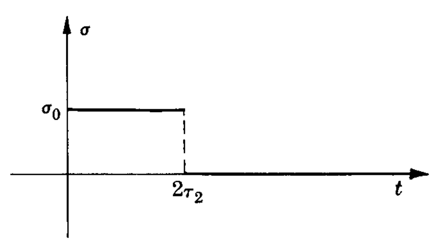


Fig. 9-15

From Problem 9.7 the response while the load is on $(t < 2\tau_2)$ is

$$\epsilon = \sigma_0[1/G_1 + (1/G_2)(1 - e^{-t/\tau_2})]$$

At $t = 2\tau_2$ the load is removed and $\sigma$ becomes zero at the same time that the "elastic" deformation $\sigma_0/G_1$ is recovered. For $t > 2\tau_2$ the response is governed by the equation $\dot\epsilon + \epsilon/\tau_2 = 0$ which is the stress-strain law for the model with $\sigma = 0$ (see Problem 9.2). The solution of this differential equation is $\epsilon = Ce^{-T/\tau_2}$ where $C$ is a constant and $T = t - 2\tau_2$. At $T = 0$, $\epsilon = C = \sigma_0(1 - e^{-2})/G_2$ and so

$$\epsilon = \sigma_0(1 - e^{-2})e^{-T/\tau_2}/G_2 = \sigma_0(e^2 - 1)e^{-t/\tau_2}/G_2 \qquad \text{for } t > 2\tau_2$$

**9.9.** The special model shown in Fig. 9-16 is elongated at a constant rate $\dot\epsilon = \epsilon_0/t_1$ as indicated in Fig. 9-17. Determine the stress in the model under this straining.

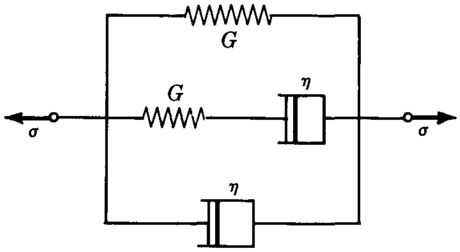



Fig. 9-16

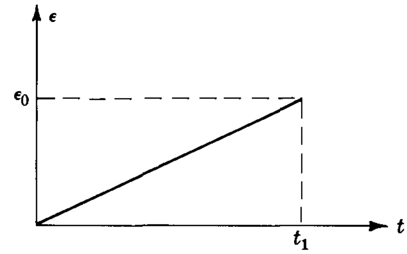



Fig. 9-17

From Problem 9.5 the stress-strain law for the model is $\dot\sigma + \sigma/\tau = \eta\ddot\epsilon + 3G\dot\epsilon + G\epsilon/\tau$ and so here $\dot\sigma + \sigma/\tau = 3G\epsilon_0/t_1 + G\epsilon_0 t/\tau t_1$. Integrating this yields $\sigma = \epsilon_0(3\eta + Gt - \eta) + Ce^{-t/\tau}$ where $C$ is the constant of integration. When $t = 0$, $\sigma = \eta\epsilon_0/t_1$ and so $C = -\eta\epsilon_0/t_1$. Thus $\sigma = \epsilon_0(2\eta + Gt - \eta e^{-t/\tau})/t_1$. Note that the same result is obtained by integrating

$$\sigma e^{t/\tau} = \frac{\epsilon_0}{t_1}\int_0^t 3Ge^{t'/\tau}\,dt' + \frac{\epsilon_0}{t_1}\int_0^t \frac{Gt'e^{t'/\tau}}{\tau}\,dt'$$

**9.10.** Determine by a direct integration of the stress-strain law for the standard linear solid its stress relaxation under the strain $\epsilon = \epsilon_0[U(t)]$.

Writing the stress-strain law (see Problem 9.2) as $\dot\sigma + (G_1 + G_2)\sigma/\eta_2 = \epsilon_0 G_1([\delta(t)] + G_1G_2[U(t)]/\eta_2)$ for the case at hand and employing the integrating factor $e^{(G_1+G_2)t/\eta_2}$ it is seen that

$$\sigma e^{(G_1+G_2)t/\eta_2} = \epsilon_0 G_1\int_0^t [\delta(t')]e^{(G_1+G_2)t'/\eta_2}\,dt' + \frac{\epsilon_0 G_1 G_2}{\eta_2}\int_0^t [U(t')]e^{(G_1+G_2)t'/\eta_2}\,dt'$$

Integrating this equation with the help of *(9.18)* and *(9.23)*,

$$\sigma = \epsilon_0 G_1(G_2 + G_1 e^{-(G_1+G_2)t/\eta_2})[U(t)]/(G_1 + G_2)$$

## CREEP AND RELAXATION FUNCTIONS. HEREDITARY INTEGRALS (Sec. 9.5)

**9.11.** Determine the relaxation function $\phi(t)$ for the three parameter model shown in Fig. 9-18.

The stress-strain relation for this model is

$$\dot\sigma + \sigma/\tau_2 = (G_1 + G_2)\dot\epsilon + G_1G_2\epsilon/\eta_2$$

and so with $\epsilon = \epsilon_0[U(t)]$ and $\dot\epsilon = \epsilon_0[\delta(t)]$ use of the integrating factor $e^{t/\tau_2}$ gives

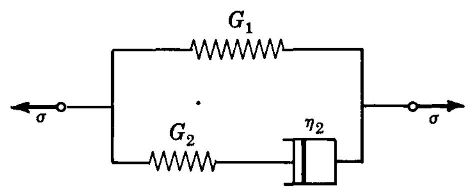



Fig. 9-18

$$\sigma e^{t/\tau_2} = \epsilon_0(G_1 + G_2)\int_0^t e^{t'/\tau_2}[\delta(t')]\,dt' + \frac{\epsilon_0 G_1 G_2}{\eta_2}\int_0^t e^{t'/\tau_2}[U(t')]\,dt'$$

Thus by use of *(9.18)* and *(9.23)*, $\sigma = \epsilon_0(G_1 + G_2 e^{-t/\tau_2}) = \epsilon_0\phi(t)$. Note that this result may also be obtained by putting $\eta_1 \to \infty$ in *(9.30)* for the generalized Maxwell model.

**9.12.** Using the relaxation function $\phi(t)$ for the model of Problem 9.11, determine the creep function by means of (*9.40*).

The Laplace transform of $\phi(t) = G_1 + G_2 e^{-t/\tau_2}$ is $\bar{\phi}(s) = G_1/s + G_2/(s+1/\tau_2)$ (see any standard table of Laplace transforms). Thus from (*9.40*),

$$\bar{\psi}(s) = (s+1/\tau_2)/[G_1 s(s+1/\tau_2) + G_2 s^2] = 1/G_1 s - [G_2/G_1(G_1+G_2)]/(s+G_1/(G_1+G_2)\tau_2)$$

which may be inverted easily by a Laplace transform table to give

$$\psi = 1/G_1 - [G_2/G_1(G_1+G_2)]e^{-G_1 t/(G_1+G_2)\tau_2}$$

This result may be readily verified by integration of the model's stress-strain equation under creep loading.

**9.13.** If a ramp type stress followed by a sustained constant stress $\sigma_1$ (Fig. 9-19) is applied to a Kelvin material, determine the resulting strain. Assume $\sigma_1/t_1 = \lambda$.

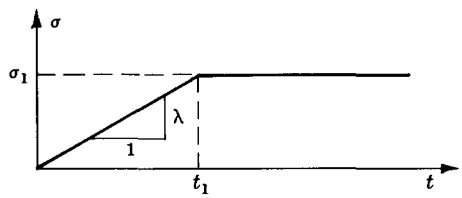


Fig. 9-19

The stress may be expressed as

$$\sigma = \lambda t[U(t)] - \lambda(t-t_1)[U(t-t_1)]$$

which when introduced into (*9.4*) leads to

$$\epsilon e^{t/\tau} = \frac{\lambda}{\eta}\left[\int_0^t t' e^{t'/\tau}[U(t')]\,dt' - \int_{t_1}^t (t'-t_1)e^{t'/\tau}[U(t'-t_1)]\,dt'\right]$$

Integrating with the aid of (*9.18*) gives

$$\epsilon = (\lambda/G)\{(t+\tau(e^{-t/\tau}-1))[U(t)] - ((t-t_1) + \tau(e^{(t_1-t)/\tau}-1))[U(t-t_1)]\}$$

which reduces as $t\to\infty$ to $\epsilon = \lambda t_1/G = \sigma_1/G$.

**9.14.** Using the creep integral (*9.34*) together with the Kelvin creep function, verify the result of Problem 9.13.

For the Kelvin body, $\psi(t) = (1-e^{-t/\tau})/G$ and (*9.34*) becomes

$$\epsilon(t) = \int_{-\infty}^t \frac{\lambda}{G}([U(t')] + t'[\delta(t')] - [U(t'-t_1)] - (t'-t_1)[\delta(t'-t_1)])(1-e^{-(t-t')/\tau})\,dt'$$

which by (*9.18*) and (*9.23*) reduces to

$$\epsilon = \frac{\lambda}{G}\left[[U(t)]\int_0^t (1-e^{-(t-t')/\tau})\,dt' - [U(t-t_1)]\int_{t_1}^t (1-e^{-(t-t')/\tau})\,dt'\right]$$

A straightforward evaluation of these integrals confirms the result presented in Problem 9.13.

**9.15.** By a direct application of the superposition principle, determine the response of a Kelvin material to the stress loading shown in Fig. 9-20.

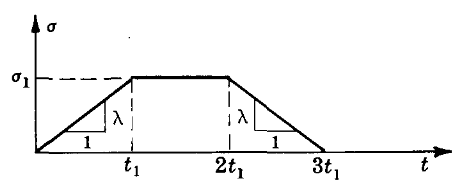


Fig. 9-20

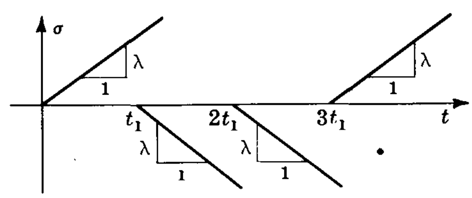


Fig. 9-21

The stress may be represented as a sequence of ramp loadings as shown in Fig. 9-21 above. From Problem 9.13, $\epsilon = (\lambda/G)[t + \tau(e^{-t/\tau} - 1)][U(t)]$ for this stress loading. In the present case therefore

$$\epsilon(t) = (\lambda/G)[(t + \tau(e^{-t/\tau} - 1))[U(t)] - ((t - t_1) + \tau(e^{-(t-t_1)/\tau} - 1))[U(t - t_1)]$$
$$-((t - 2t_1) + \tau(e^{-(t-2t_1)/\tau} - 1))[U(t - 2t_1)] + ((t - 3t_1) + \tau(e^{-(t-3t_1)/\tau} - 1))[U(t - 3t_1)]]$$

Note that as $t \to \infty$, $\epsilon \to 0$.

## COMPLEX MODULI AND COMPLIANCES (Sec. 9.6)

**9.16.** Determine the complex modulus $G^*$ and the lag angle $\delta$ for the Maxwell material of Fig. 9-2.

Writing (9.3) as $\dot{\sigma} + \sigma/\tau = G\dot{\epsilon}$ and inserting (9.41) and (9.42) gives $i\omega\sigma_0 e^{i\omega t} + \sigma_0 e^{i\omega t}/\tau = Gi\omega\epsilon_0 e^{i(\omega t - \delta)}$ from which $\sigma_0 e^{i\delta}/\epsilon_0 = G^* = Gi\omega\tau/(1 + i\omega\tau)$, or in standard form

$$G^* = G(\omega^2\tau^2 + i\omega\tau)/(1 + \omega^2\tau^2)$$

From Fig. 9-11, $\tan\delta = G_2/G_1 = G\omega\tau/G\omega^2\tau^2 = 1/\omega\tau$.

**9.17.** Show that the result of Problem 9.16 may also be obtained by simply replacing the operator $\partial_t$ by $i\omega$ in equation (9.5) and defining $\sigma/\epsilon = G^*$.

After the suggested substitution (9.5) becomes $(i\omega/G + 1/\eta)\sigma = i\omega\epsilon$ from which

$$\sigma/\epsilon = Gi\omega/(i\omega + 1/\tau) = Gi\omega\tau/(1 + i\omega\tau)$$

as before.

**9.18.** Use equation (9.10) for the generalized Maxwell model to illustrate the rule that "for models in parallel, the complex moduli add".

From Problem 9.17 the complex modulus for the Maxwell model may be written $G^* = \sigma/\epsilon = Gi\omega\tau/(1 + i\omega\tau)$. Thus writing (9.10) as

$$\sigma = G_1\{\partial_t\}\epsilon/\{\partial_t + 1/\tau_1\} + G_2\{\partial_t\}\epsilon/\{\partial_t + 1/\tau_2\} + \cdots + G_N\{\partial_t\}\epsilon/\{\partial_t + 1/\tau_N\}$$

the generalized Maxwell complex modulus becomes

$$G^* = G_1 i\omega\tau_1/(1 + i\omega\tau_1) + G_2 i\omega\tau_2/(1 + i\omega\tau_2) + \cdots + G_N i\omega\tau_N/(1 + i\omega\tau_N) = G_1^* + G_2^* + \cdots + G_N^*$$

**9.19.** Verify the relationship $J_1 = 1/G_1(1 + \tan^2\delta)$ between the storage modulus and compliance.

From (9.43) and (9.44), $J^* = 1/G^*$ and so $J_1 - iJ_2 = 1/(G_1 + iG_2) = (G_1 - iG_2)/(G_1^2 + G_2^2)$. Thus

$$J_1 = G_1/(G_1^2 + G_2^2) = 1/G_1(1 + (G_2/G_1)^2) = 1/G_1(1 + \tan^2\delta)$$

**9.20.** Show that the energy dissipated per cycle is related directly to the loss compliance $J_2$ by evaluating the integral $\int \sigma\, d\epsilon$ over one cycle.

For the stress and strain vectors of Fig. 9-10, the integral $\int \sigma\, d\epsilon$ evaluated over one cycle is

$$\int_0^{2\pi/\omega} \sigma \frac{d\epsilon}{dt}\, dt = \int_0^{2\pi/\omega} (\sigma_0 \sin\omega t)\epsilon_0\omega \cos(\omega t - \delta)\, dt$$
$$= \sigma_0\epsilon_0\omega \int_0^{2\pi/\omega} \sin\omega t\,(\cos\omega t \cos\delta + \sin\omega t \sin\delta)\, dt$$
$$= \sigma_0^2\omega\left[J_1 \int_0^{2\pi/\omega} \frac{\sin 2\omega t}{2}\, dt + J_2 \int_0^{2\pi/\omega} (\sin^2\omega t)\, dt\right] = \sigma_0^2\pi J_2$$

## THREE DIMENSIONAL THEORY. VISCOELASTIC STRESS ANALYSIS (Sec. 9.7-9.8)

**9.21.** Combine (*9.48a*) and (*9.48b*) to obtain the viscoelastic constitutive relation $\sigma_{ij} = \delta_{ij}\{R\}\epsilon_{kk} + \{S\}\epsilon_{ij}$ and determine the form of the operators $\{R\}$ and $\{S\}$.

Writing (*9.48a*) as $\{P\}(\sigma_{ij} - \delta_{ij}\sigma_{kk}/3) = 2\{Q\}(\epsilon_{ij} - \delta_{ij}\epsilon_{kk}/3)$ and replacing $\sigma_{kk}$ here by the right hand side of (*9.48b*), the result after some simple manipulations is

$$\sigma_{ij} = \delta_{ij}\{(3KP - 2Q)/3P\}\epsilon_{kk} + \{2Q/P\}\epsilon_{ij}$$

**9.22.** A bar made of Kelvin material is pulled in tension so that $\sigma_{11} = \sigma_0[U(t)]$, $\sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{23} = \sigma_{31} = 0$ where $\sigma_0$ is constant. Determine the strain $\epsilon_{11}$ for this loading.

From (*9.48b*), $3\epsilon_{ii} = \sigma_0[U(t)]/K$ for this case; and from (*9.48a*) with $i = j = 1$, $\{P\}(\sigma_{11} - \sigma_{11}/3) = \{2Q\}(\epsilon_{11} - \epsilon_{ii}/3)$. But from (*9.6*), $\{P\} = 1$ and $\{Q\} = \{G + \eta\partial_t\}$ for a Kelvin material; so that now

$$2\sigma_0[U(t)]/3 = 2\{G + \eta\partial_t\}(\epsilon_{11} - \sigma_0[U(t)]/9K)$$

or

$$\dot\epsilon_{11} + \epsilon_{11}/\tau = \sigma_0[U(t)](3K + G)/9\eta K + \sigma_0[\delta(t)]/9K$$

Solving this differential equation yields

$$\epsilon_{11} = \sigma_0(3K + G)(1 - e^{-t/\tau})[U(t)]/9KG + \sigma_0 e^{-t/\tau}[U(t)]/9K$$

As $t \to \infty$, $\epsilon_{11} \to (3K + G)\sigma_0/9KG = \sigma_0/E$.

**9.23.** A block of Kelvin material is held in a container with rigid walls so that $\epsilon_{22} = \epsilon_{33} = 0$ when the stress $\sigma_{11} = -\sigma_0[U(t)]$ is applied. Determine $\epsilon_{11}$ and the retaining stress components $\sigma_{22}$ and $\sigma_{33}$ for this situation.

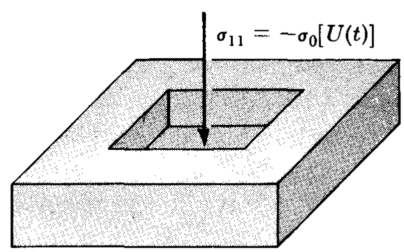



Fig. 9-22

Here $\epsilon_{ii} = \epsilon_{11}$ and $\sigma_{22} = \sigma_{33}$ so that (*9.48b*) becomes $\sigma_{11} + 2\sigma_{22} = 3K\epsilon_{11}$ and (*9.48a*) gives $2(\sigma_{11} - \sigma_{22})/3 = 2G\{1 + \tau\partial_t\}(2\epsilon_{11}/3)$ for a Kelvin body. Combining these relations yields the differential equation

$$\dot\epsilon_{11} + (4G + 3K)\epsilon_{11}/4G\tau = -3\sigma_0[U(t)]/4G\tau$$

which upon integration gives

$$\epsilon_{11} = -3\sigma_0[U(t)](1 - e^{-(4G+3K)t/4G\tau})/(4G + 3K)$$

Inserting this result into (*9.48a*) for $i = j = 2$ gives

$$\sigma_{22} = (+\sigma_0/2 - 9K\sigma_0(1 - e^{-(4G+3K)t/4G\tau})/(8G + 6K))[U(t)]$$

**9.24.** The radial stress component in an elastic half-space under a concentrated load at the origin may be expressed as

$$\sigma_{(rr)} = (P/2\pi)[(1 - 2\nu)\alpha(r, z) - \beta(r, z)]$$

where $\alpha$ and $\beta$ are known functions. Determine the radial stress for a Kelvin viscoelastic half-space by means of the correspondence principle when $P = P_0[U(t)]$.

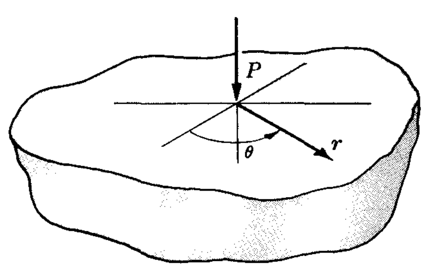



Fig. 9-23

The viscoelastic operator for the term $(1-2\nu)$ is $\{3Q\}/\{3KP+Q\}$ so that for a Kelvin body the transformed viscoelastic solution becomes

$$\bar{\sigma}_{(rr)} = \frac{3P_0}{2\pi s}\left[\frac{G+\eta s}{3K+G+\eta s}\,\alpha(r,z) - \beta(r,z)\right]$$

which may be inverted with help of partial fractions and transform tables to give the viscoelastic stress

$$\sigma_{(rr)} = \frac{3P_0}{2\pi}\left[\left(\frac{G}{3K+G} + \frac{3K}{3K+G}\,e^{-(3K+G)t/\eta}\right)\alpha(r,z) + \beta(r,z)\right]$$

**9.25.** The correspondence principle may be used to obtain displacements as well as stresses. The $z$ displacement of the surface of the elastic half space in Problem 9.24 is given by $w_{(z=0)} = P(1-\nu^2)/E\pi r$. Determine the viscoelastic displacement of the surface for the viscoelastic material of that problem.

The viscoelastic operator corresponding to $(1-\nu^2)/E$ is $\{3K+4Q\}/4Q(3K+Q)$ which for the Kelvin body causes the transformed displacement to be

$$\bar{w}_{(z=0)} = P_0(3K+4(G+\eta s))/4\pi rs(3K+G+\eta s)(G+\eta s)$$

After considerable manipulation and inverting, the result is

$$w_{(z=0)} = \frac{P_0(3K+4G)}{4\pi r^2(3K+G)}\left[\frac{1}{G} - \frac{3e^{-(3K+G)t/\eta}}{3K+4G} - \frac{3K+G}{G(3K+4G)}\,e^{-t/\tau}\right]$$

Note that when $t=0$, $w_{(z=0)}=0$ and when $t\to\infty$, $w_{(z=0)}\to P_0(1-\nu^2)/E\pi r$, the elastic deflection.

**9.26.** A simply supported uniformly loaded beam is assumed to be made of a Maxwell material. Determine the bending stress $\sigma_{11}$ and the deflection $w(x_1,t)$ if the load is $p = p_0[U(t)]$.

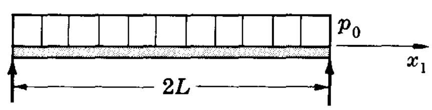



Fig. 9-24

The bending stress for a simply supported elastic beam does not depend upon material properties, so the elastic and viscoelastic bending stress here are the same. The elastic deflection of the beam is $w(x_1) = p_0\alpha(x_1)/24EI$ where $\alpha(x_1)$ is a known function. For a Maxwell body, $\{P\} = \{\partial_t + 1/\tau\}$ and $\{Q\} = \{G\partial_t\}$, so that the transformed deflection is

$$\bar{w} = \frac{p_0\alpha(x_1)}{24I}\left(\frac{3K/\tau + (3K+G)s}{9KGs^2}\right)$$

which when inverted gives

$$w(x_1,t) = \frac{p_0\alpha(x_1)}{24I}\left(\frac{t}{3\eta} + \frac{3K+G}{9KG}\right)$$

When $t=0$, $w(x_1,0) = p_0\alpha(x_1)/24EI$, the elastic deflection.

**9.27.** Show that as $t\to\infty$ the stress $\sigma_{22}$ in Problem 9.23 approaches $\sigma_0$ (material behaves as a fluid) if the material is considered incompressible $(\nu = 1/2)$.

From Problem 9.23,

$$\sigma_{22}|_{t\to\infty} = -\sigma_0(9K-(4G+3K))/2(4G+3K) = -\sigma_0(3K-2G)/(3K+4G)$$

which may be written in terms of $\nu$ as $\sigma_{22}|_{t\to\infty} = -\nu\sigma_0/(1-\nu)$. Thus for $\nu = 1/2$, $\sigma_{22}|_{t\to\infty} = -\sigma_0$.

## MISCELLANEOUS PROBLEMS

**9.28.** Determine the constitutive relation for the Kelvin-Maxwell type model shown in Fig. 9-25 and deduce from the result the Kelvin and Maxwell stress-strain laws.

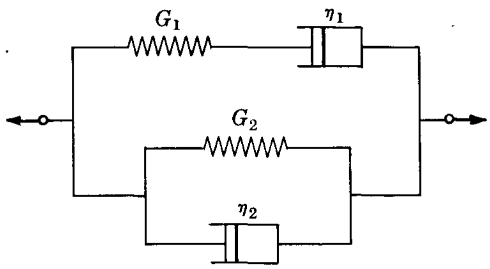



Fig. 9-25

Here

$$\sigma = \sigma_M + \sigma_K = \dot{\epsilon}/\{\partial_t/G_1 + 1/\eta_1\} + \{G_2 + \eta_2\partial_t\}\epsilon$$

which upon application of the time operators becomes

$$\dot{\sigma} + \sigma/\tau_1 = \eta_2\ddot{\epsilon} + (G_1 + G_2 + \eta_2/\tau_1)\dot{\epsilon} + (G_2/\tau_1)\epsilon$$

In this equation if $\eta_2 = 0$ (spring in parallel with Maxwell), $\dot{\sigma} + \sigma/\tau_1 = (G_1 + G_2)\dot{\epsilon} + (G_2/\tau_1)\epsilon$. Further, if $G_2 = 0$, the Maxwell law $\dot{\sigma} + \sigma/\tau_1 = G_1\dot{\epsilon}$ results. Likewise, if $G_2$ is taken zero first (dashpot in parallel with Maxwell), $\dot{\sigma} + \sigma/\tau_1 = \eta_2\ddot{\epsilon} + (G_1 + \eta_2/\tau_1)\dot{\epsilon}$; and when $\eta_2 = 0$, this also reduces to the Maxwell law.

If the four-parameter constitutive relation is rewritten

$$\eta_1\dot{\sigma} + G_1\sigma = \eta_1\eta_2\ddot{\epsilon} + (G_1\eta_1 + G_2\eta_1 + G_1\eta_2)\dot{\epsilon} + G_1G_2\epsilon$$

and $\eta_1$ set equal to zero, the result is the Kelvin law $\sigma = \eta_2\dot{\epsilon} + G_2\epsilon$. Likewise, if $G_1 = 0$ the reduced equation is $\dot{\sigma} = \eta_2\ddot{\epsilon} + G_2\dot{\epsilon}$, again representing the Kelvin model.

**9.29.** Use the superposition principle to obtain the creep recovery response for the standard linear solid of Fig. 9-3(*a*) and compare the result with that obtained in Problem 9.8.

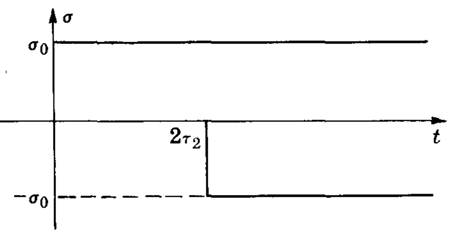



Fig. 9-26

With the stress loading expressed by

$$\sigma = \sigma_0[U(t)] - \sigma_0[U(t - 2\tau_2)]$$

(see Fig. 9-26), the strain may be written at once from the result of Problem 9.7 as

$$\epsilon = \sigma_0(1/G_1 + (1 - e^{-t/\tau_2})/G_2)[U(t)] - \sigma_0(1/G_1 + (1 - e^{-(t-2\tau_2)\tau_2})/G_2)[U(t - 2\tau_2)]$$

At times $t > 2\tau_2$ both step functions equal unity and

$$\epsilon = \sigma_0(-e^{-t/\tau_2} + e^{-(t-2\tau_2)/\tau_2})/G_2 = \sigma_0(e^2 - 1)e^{-t/\tau_2}/G_2$$

which agrees with the result in Problem 9.8.

**9.30.** Determine the stress in the model of Problem 9.9 when subjected to the strain history shown in Fig. 9-27. Show that eventually the "free" spring in the model carries the entire stress.

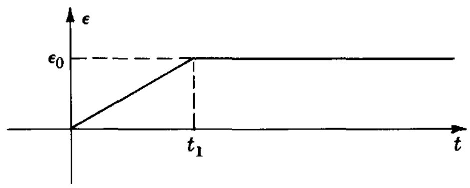



Fig. 9-27

From Problem 9.9 and the superposition principle the stress is

$$\sigma = \epsilon_0(\eta(2 - e^{-t/\tau}) + Gt)[U(t)]/t_1 - \epsilon_0(\eta(2 - e^{-(t-t_1)/\tau}) + G(t - t_1))[U(t - t_1)]/t_1$$

For times $t > t_1$ the stress is $\sigma = \epsilon_0\eta(e^{t_1/\tau} - 1)e^{-t/\tau}/t_1 + G\epsilon_0$, and as $t \to \infty$ this reduces to $\sigma = G\epsilon_0$.

**9.31.** The "logarithmic retardation spectrum" $L$ is defined in terms of the retardation spectrum $J$ by $L(\ln \tau) = \tau J(\tau)$. From this definition determine the creep function $\psi(t)$ in terms of $L(\ln \tau)$.

Let $\ln \tau = \lambda$ so that $e^\lambda = \tau$ and thus $d\tau/d\lambda = e^\lambda = \tau$, or $d\tau = \tau d(\ln \tau)$. From this, (*9.28*) defining $\psi(t)$ becomes $\psi(t) = \int_0^\infty L(\ln \tau)(1 - e^{-t/\tau})\, d(\ln \tau)$. In the same way, if $H(\ln \tau) = \tau G(\tau)$ defines the logarithmic relaxation spectrum, $\phi(t)$ in (*9.31*) may be written

$$\phi(t) = \int_0^\infty H(\ln \tau) e^{-t/\tau} d(\ln \tau)$$

**9.32.** For the Maxwell model of Fig. 9-2(*a*) determine the storage and loss moduli, $G_1$ and $G_2$, as functions of $\ln \omega\tau$ and sketch the shape of these functions.

From Problem 9.16,

$$G^* = G(\omega^2\tau^2 + i\omega\tau)/(1 + \omega^2\tau^2)$$

for a Maxwell material. Thus

$$G_1 = G\omega^2\tau^2/(1 + \omega^2\tau^2) = Ge^{2\lambda}/(1 + e^{2\lambda})$$

where $\lambda = \ln \omega\tau$. For $\lambda = 0$, $G_1 = G/2$; for $\lambda = \infty$, $G_1 = G$; and for $\lambda = -\infty$, $G_1 = 0$. Likewise $G_2 = Ge^\lambda/(1 + e^{2\lambda})$ and for $\lambda = 0$, $G_2 = G/2$; for $\lambda = \pm\infty$, $G_2 = 0$. The shape of the curves for these functions is as shown in Fig. 9-28.

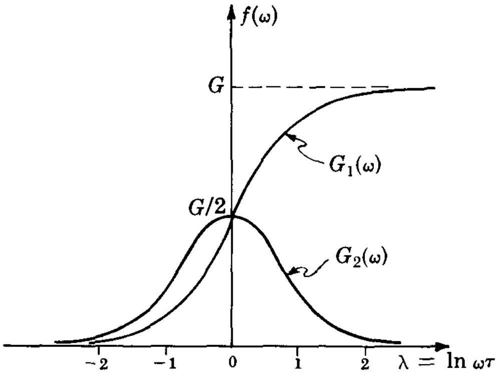



Fig. 9-28

**9.33.** Determine the viscoelastic operator form of the elastic constant $\nu$ (Poisson's ratio) using the constitutive relations (*9.48*).

Under a uniaxial tension $\sigma_{11} = \sigma_0$, (*9.48b*) gives $\epsilon_{ii}/3 = \sigma_0/9K$ so that (*9.48a*) for $i = j = 1$ yields $\epsilon_{11} = \{3KP + Q\}\sigma_0/\{9KQ\}$. In the same way (*9.48a*) for $i = j = 2$ yields $\epsilon_{22} = \{2Q - 3PK\}\sigma_0/\{18KQ\}$. Thus in operator form, $\nu = -\epsilon_{22}/\epsilon_{11} = \{3PK - 2Q\}/\{6KP + 2Q\}$.

**9.34.** A cylindrical viscoelastic body is inserted into a rigid snug-fitting container (Fig. 9-29) so that $\epsilon_{(rr)} = 0$ (no radial strain). The body is elastic in dilatation and has the creep function $\psi_s = A + Bt + Ce^{\lambda t}$ where $A, B, C, \lambda$ are constants. If $\dot\epsilon_{33} = \dot\epsilon_0[U(t)]$, determine $\sigma_{33}(t)$.

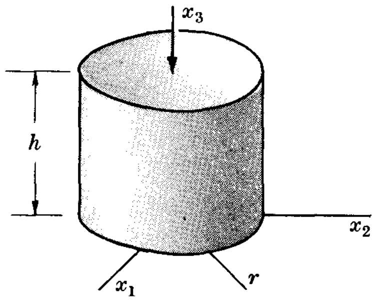



Fig. 9-29

Here $\sigma_{ii} = 3K\epsilon_{ii}$ and by the symmetry of the problem, $2\sigma_{11} + \sigma_{33} = 3K\epsilon_{33}$. Also from (*9.50a*) with $i = j = 1$, $\sigma_{11} - \sigma_{33} = -\int_0^t \frac{d\epsilon_{33}}{dt'} \phi_s(t - t')\, dt'$. Solving for $\sigma_{33}$ from these two relations we obtain

$$\sigma_{33} = K\epsilon_{33} + \frac{2}{3}\int_0^t \frac{d\epsilon_{33}}{dt'} \phi_s(t - t')\, dt'$$

The relaxation function $\phi_s$ may be found with the help of (*9.40*). The result is

$$\phi_s = [(r_1 - \lambda)e^{r_1 t} - (r_2 - \lambda)e^{r_2 t}]/(r_1 - r_2)$$

where $r_{1,2} = [A\lambda - B \pm \sqrt{(A\lambda + B)^2 + 4BC\lambda}\,]/2(A + C)$. Thus finally

$$\sigma_{33} = K\dot{\epsilon}_0 t + \frac{2}{3}\int_0^t \dot{\epsilon}_0 \frac{[(r_1-\lambda)e^{r_1(t-t')} - (r_2-\lambda)e^{r_2(t-t')}]}{(r_1-r_2)}[U(t')]\,dt'$$

which upon integration gives

$$\sigma_{33} = (K\dot{\epsilon}_0 t + 2\dot{\epsilon}_0[-(r_1-\lambda)(1-e^{r_1 t})/3r_1 + (r_2-\lambda)(1-e^{r_2 t})/3r_2]/(r_1-r_2))[U(t)]$$

**9.35.** The "creep buckling" of a viscoelastic column may be analyzed within the linear theory through the correspondence principle. Determine the deflection $w(x_1, t)$ of a Kelvin pinned-end column by this method.

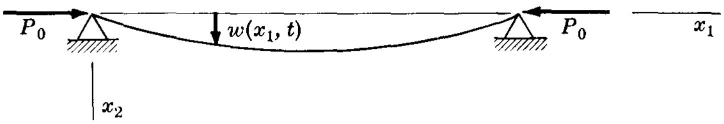



Fig. 9-30

The elastic column formula is $d^2w/dx_1^2 + P_0 w/EI = 0$, and for a Kelvin material $E$ may be replaced by the operator $\{E + \eta\partial_t\}$ so that for the viscoelastic column $\{E + \eta\partial_t\}(d^2w/dx_1^2) + P_0 w/I = 0$. Assuming the deflection in a product form $w(x_1, t) = W(x_1)\theta(t)$, the operator leads to the differential equation

$$(E\theta + \eta\dot{\theta})(d^2W/dx_1^2) + P_0 W\theta/I = 0 \quad \text{from which} \quad \dot{\theta} + [1 + P_0 W/EI(d^2W/dx_1^2)]\theta/\tau = 0$$

where $\tau = \eta/E$. But the elastic buckling load is $P_B = -EI(d^2W/dx_1^2)/W$ and so $\dot{\theta} + (1 - P_0/P_B)\theta/\tau = 0$ which integrates easily to yield $\theta = e^{(P_0/P_B - 1)t/\tau}$. Finally then the "creep buckling" deflection is $w = We^{(P_0/P_B-1)t/\tau}$.

**9.36.** Formulate the steady-state vibration problem for a viscoelastic beam assuming the constitutive relations are those given by (*9.48*).

The free vibrations of an elastic beam are governed by the equation $EI(\partial^4 w/\partial x_1^4) + \rho A(\partial^2 w/\partial t^2) = 0$. From (*9.48*) the viscoelastic operator for $E$ is $\{9KQ/(3KP + Q)\}$, and if the deflection $w(x_1, t) = W(x_1)\theta(t)$ the resulting viscoelastic differential equation may be split into the space equation $d^4W/dx_1^4 - k^4 W = 0$ and the time equation $\{3KP + Q\}(d^2\theta/dt^2) + (k^4 I/\rho A)\{9KQ\}(\theta) = 0$. The solution $W_i$ of the space equation represents the $i$th mode shape, and from the time equation for $k = k_i$ the solution $\theta_i = \sum_{j=1}^{N} A_{ij}e^{\lambda_{ij}t}$ where $N$ depends upon the degree of the operator. The total solution therefore is $w(x_1, t) = \sum_{i=1}^{\infty}\sum_{j=1}^{N} W_i(x_1)A_{ij}e^{\lambda_{ij}t}$ in which the $\lambda_{ij}$ are complex.

## Supplementary Problems

9.37. Determine the constitutive equation for the four parameter model shown in Fig. 9-31.
*Ans.* $\ddot{\sigma} + (G_1/\eta_2 + G_2/\eta_2 + G_1/\eta_1)\dot{\sigma} + (G_1G_2/\eta_1\eta_2)\sigma = G_1\ddot{\epsilon} + (G_1G_2/\eta_2)\dot{\epsilon}$

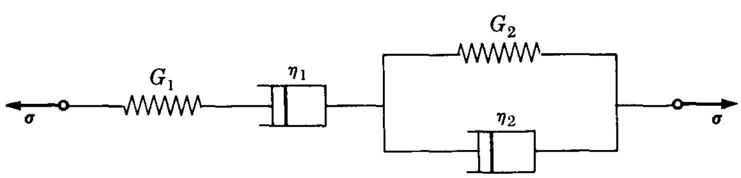



Fig. 9-31

9.38. Determine the creep response of the standard linear solid by direct integration of $\dot{\epsilon} + \epsilon/\tau_2 = \sigma_0[U(t)](G_1 + G_2)/G_1\eta_2 + \sigma_0[\delta(t)]/G_1$. (See Problem 9.7.)

9.39. Deduce the Kelvin and Maxwell stress-strain laws from the results established in Problem 9.5 for the four parameter model of that problem. (*Hint.* Let $G_3 = 0$, etc.)

9.40. Use equation (*9.40*) to obtain $\psi(t)$ if $\phi(t) = a(b/t)^m$ with $m < 1$. (*Hint.* Take $m = 1 - k$; then $\phi(t) = ab^m t^{k-1}$.) *Ans.* $\psi(t) = \dfrac{\sin \pi m}{am\pi}\left(\dfrac{t}{b}\right)^m$

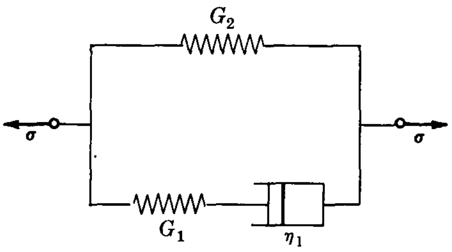


Fig. 9-32

9.41. Determine the creep and relaxation functions for the model shown in Fig. 9-32.
*Ans.* $\psi(t) = 1/G_2 - G_1 e^{-G_2 t/(G_1+G_2)\tau_1}/G_2(G_1 + G_2)$
$\phi(t) = G_2 + G_1 e^{-t/\tau_1}$

9.42. Determine $G^*$ for the model shown in Fig. 9-33.

*Ans.* $$G^* = \frac{G_1(1 + \tau_2^2\omega^2) + G_2\omega^2\tau_2^2}{1 + \omega^2\tau_2^2} + i\,\frac{\omega(G_2\tau_2 + \eta_3(1 + \tau_2^2\omega^2))}{1 + \omega^2\tau_2^2}$$

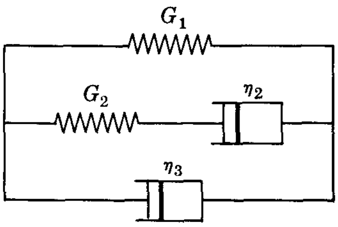


Fig. 9-33

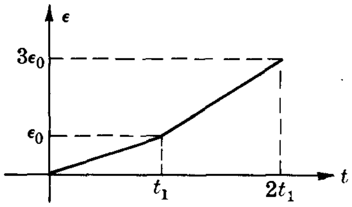


Fig. 9-34

9.43. In the model of Problem 9.42 let $G_1 = G_2 = G$ and $\eta_2 = \eta_3 = \eta$ and determine the stress history of the resulting model when it is subjected to the strain sequence shown in Fig. 9-34.

*Ans.* $\sigma = \dfrac{\epsilon_0}{t_1}\,(G(2t - t_1) + \eta(4 - (1 + e^{t_1/\tau})e^{-t/\tau}))$ for $t_1 < t < 2t_1$

9.44. A viscoelastic block having the constitutive equation $\dot{\sigma} + \alpha\sigma = \beta\dot{\epsilon} + \gamma\epsilon$ where $\alpha, \beta, \gamma$ are constants is loaded under conditions such that $\sigma_{11} = -\sigma_0[U(t)]$, $\sigma_{22} = 0$, $\epsilon_{33} = 0$ (see Fig. 9-35). Assuming $\sigma_{ii} = 3K\epsilon_{ii}$, determine $\sigma_{33}(t)$, $\sigma_{33}(0)$ and $\sigma_{33}(\infty)$.

*Ans.* $$\sigma_{33} = -\sigma_0\left[\frac{3\alpha K - 2\gamma}{2(3\alpha K + \gamma)\lambda} + \left(\frac{3K - 2\beta}{2(3K + \beta)} - \frac{3\alpha K - 2\gamma}{2(3\alpha K + \gamma)\lambda}\right)e^{-\lambda t}\right]$$ where $\lambda = (3\alpha K + \gamma)/(3K + \beta)$.

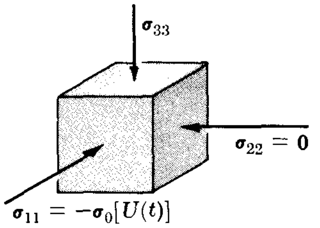


Fig. 9-35

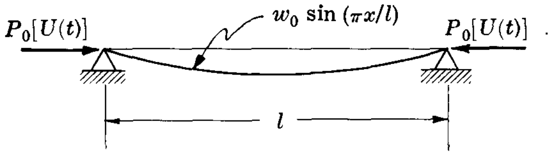


Fig. 9-36

9.45. A pinned-end viscoelastic column is a Maxwell material for which $\dot{\sigma} + \sigma/\tau = E\dot{\epsilon}$. The initial shape of the column is $w = w_0 \sin(\pi x/l)$ when the load $P_0[U(t)]$ is applied (see Fig. 9-36). Determine the subsequent deflection $w(x_1, t)$ as a function of $P_B$, the elastic buckling load.
*Ans.* $w(x_1, t) = w_0 \sin(\pi x_1/l)e^{-t/(1-P_B/P_0)\tau}$

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